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Adjacency Matrix of Product of Graphs

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Abstract

In graph theory, different types of matrices associated with graph, e.g. Adjacency matrix, Incidence matrix, Laplacian matrix etc. Among all adjacency matrix play an important role in graph theory. Many products of two graphs as well as its generalized form had been studied, e.g., cartesian product, 2−cartesian product, tensor product, 2−tensor product etc. In this paper, we discuss the adjacency matrix of two new product of graphs $G \maltese H$, where $\maltese = \otimes_2, \times_2$. Also, we obtain the spectrum of these products of graphs.

Keywords:- 2−cartesian product, 2−tensor product, adjacency matrix and spectrum

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1 Introduction

Graph theory, no doubt, is the fast growing area of combinatory. Because of its inherent simplicity. Graph theory has a wide range of applications in Computer Science, Sociology, bio-informatics, medicines etc. There are different type of product of graphs e.g., cartesian product, tensor product etc. defined in graph theory. There are many matrices associated with graphs like adjacency matrix, incidence matrix, Laplacian matrix etc. One of them is adjacency matrix. It is well- known that these product operation on graphs and product of adjacency matrices are related ([6], [9]).

We have generalized well-known two products, cartesian product and tensor product with the help of concept of distance. In this direction, we have defined 2−cartesian product $G \times_2 H$ and 2−tensor product $G \otimes_2 H$ of graphs G and H ([1], [2], [3]).

Let $G = (V(G), E(G))$ be a finite and simple graph. For a connected graph G, $d_G(u, u')$ is the length of the shortest path between u and u' in G. A graph is $r-$regular if every vertex of G has degree r. For the basic terminology, concepts and results of graph theory, we refer to ([5], [8]).

The Kronecker product $A \otimes B$ of two matrices A and B has been defined and discussed in [6]

Definition 1.1 Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be in $M_{m\times n}(\mathbb{R})$. Then the Kronecker product of A and B, denoted by $A \otimes B$ is defined as the partitioned matrix $[a_{ij}B]$,

$$A \otimes B = [a_{ij}B] = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}$$

Note that it has mn blocks, where ij^{th} block is the block matrix $a_{ij}B$ of order $m \times n$.

Let $X = [x_1, x_2, \ldots, x_n]^T$ and $Y = [y_1, y_2, \ldots, y_n]^T$ be column vectors. Then by definition of the kronecker product, we have

$$X \otimes Y = \begin{bmatrix} x_1Y & x_2Y & \ldots & x_nY \end{bmatrix}^T = \begin{bmatrix} x_1y_1 & \ldots & x_1y_m & \ldots & x_ny_1 & \ldots & x_ny_m \end{bmatrix}^T.$$

Definition 1.2 [6] Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be in $M_{m\times n}(\mathbb{R})$. Then the Kronecker sum of A and B, denoted by $A \dagger B$ and defined as $A \dagger B = (A \otimes I_m) + (I_n \otimes B)$.

Definition 1.3 [9] Let $G = (V, E)$ be a simple graph with the vertex set $\{u_1, u_2, \ldots u_n\}$. Then the adjacency matrix $A(G) = [a_{ij}]$ is defined as follows:

$$a_{ij} \quad = \quad \begin{cases} 1; & \text{if } d_G(u_i, u_j) = 1 \\ 0; & \text{otherwise.} \end{cases}$$

Then $A(G)$ is a real, square symmetric matrix.

Next, we recall the definitions of $2-$tensor product and $2-$cartesian product of graphs as follows:

Definition 1.4 [2] Let $G = (U, E_1)$ and $H = (V, E_2)$ be two connected graphs. The $2-$tensor product $G \otimes_2 H$ of G and H is the graph with vertex set $U \times V$ and two vertices (u, v) and (u', v') in $V(G \otimes_2 H)$ are adjacent in $2-$tensor product if $d_G(u, u') = 2$ and $d_H(v, v') = 2$.
Note that if $d_G(u, u') = 1 = d_H(v, v')$, then it is the usual tensor product $G \otimes H$ of G and H.

Definition 1.5 [3] Let $G = (U, E_1)$ and $H = (V, E_2)$ be two connected graphs. The $2-$cartesian product $G \times_2 H$ of G and H is the graph with vertex set $U \times V$ and two vertices (u, v) and (u', v') in $V(G \times_2 H)$ are adjacent if one of the following conditions is satisfied:

(i) $d_G(u, u') = 2$ and $d_H(v, v') = 0$,

(ii) $d_G(u, u') = 0$ and $d_H(v, v') = 2$.

Note that in the above condition (i) and condition (ii), 2 replace by 1, then it is the usual cartesian product $G \times H$ of G and H.

It is clear that the definition of $A(G)$ is in terms of adjacent vertices, i.e., vertices at distance 1. For the graphs $G✠H$, we use $2-$distance between vertices. So, we require to consider the second stage adjacency matrix.

Definition 1.6 [4] Let $G = (V, E)$ be a simple graph with the vertex set $\{u_1, u_2, \ldots u_n\}$. The second stage adjacency matrix $A_2(G) = [a_{ij}]$ is defined as follows:

$$a_{ij} = \begin{cases} 1; & \text{if } d_G(u_i, u_j) = 2 \\ 0; & \text{otherwise.} \end{cases}$$

Note that $A_2(G)$ is a real, square symmetric matrix.

2 Adjacency matrix of $G \maltese H$

In this section, we discuss adjacency matrix of $G \otimes_2 H$ and $G \times_2 H$ of graphs G and H.

Adjacency matrix of usual tensor product and cartesian product of graphs can be obtained as follows.

Proposition 2.1 [9] Let G and H be simple graphs with n and m vertices respectively. Then

(i) $A(G \otimes H) = A(G) \otimes A(H)$.

(ii) $A(G \times H) = (A(G) \otimes I_m) + (I_n \otimes A(H))$.

Next, we prove the result similar to Proposition 2.1 for $G \otimes_2 H$ and $G \times_2 H$.

Proposition 2.2 Let G and H be connected graphs with n and m vertices respectively. Then

$$A(G \otimes_2 H) = A_2(G) \otimes A_2(H)$$

. **Proof:** Let $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(H) = \{v_1, v_2, \ldots, v_m\}$.

Let $A(G \otimes_2 H) = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1s} & \cdots & A_{1n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{rs} & \cdots & A_{rn} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{ns} & \cdots & A_{nn} \end{bmatrix}$

Where

$$A_{r,s} = \begin{array}{c} \\ (u_r, v_1) \\ \vdots \\ (u_r, v_m) \end{array} \begin{array}{c} \begin{array}{cccc} (u_s, v_1) & (u_s, v_2) & \cdots & (u_s, v_m) \end{array} \\ \begin{bmatrix} P_{11} & P_{12} & \cdots & P_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ P_{m1} & P_{m2} & \cdots & P_{mm} \end{bmatrix} \end{array}$$

Suppose $d_G(u_r, u_s) \neq 2$, then A_{rs} is a zero matrix of order $m \times m$. Suppose $d_G(u_r, u_s) = 2$, then

$$P_{ij} = \begin{cases} 1; & \text{if } d_H(v_i, v_j) = 2 \\ 0; & \text{otherwise.} \end{cases}$$

Thus, in this case, $P_{ij} = (ij)^{th}$entry of $A_2(H)$. So,

$$A_{rs} = \begin{cases} A_2(H); & \text{if } d_G(u_r, u_s) = 2 \\ 0; & \text{otherwise.} \end{cases}$$

Now let $A_2(G) = [b_{ij}]_{n\times n}$ and $A_2(H) = [c_{ij}]_{m\times m}$. Then

$$A_2(G) \otimes A_2(H) = \begin{bmatrix} b_{11}A_2(H) & \cdots & b_{1n}A_2(H) \\ \vdots & \ddots & \vdots \\ b_{n1}A_2(H) & \cdots & b_{nn}A_2(H) \end{bmatrix}$$

So, $(ij)^{th}$ block of $A_2(G) \otimes A_2(H)$ is the block matrix $b_{ij}A_2(H)$ of order $m \times m$. Then the

block matrix

$$[b_{ij}A_2(H)] = \begin{cases} A_2(H); & if\ b_{ij} = 1,\ i.e.,\ d_G(u_i, u_j) = 2 \\ 0_{m\times m}; & \text{otherwise.} \end{cases}$$

$$= [A_{ij}]$$

Therefore, $[b_{ij}A_2(H)] = (ij)^{th}$ block of $A(G \otimes_2 H)$. So, $A(G \otimes_2 H) = A_2(G) \otimes A_2(H)$.

Proposition 2.3 Let G and H be connected graphs with n and m vertices respectively. Then $A(G \times_2 H) = [A_2(G) \otimes I_m] + [I_n \otimes A_2(H)]$.

Proof: Suppose $V(G) = \{u_1, u_2, \ldots, u_n\}$ and $V(H) = \{v_1, v_2, \ldots, v_n\}$. Then consider $A(G \times_2 H)$ and A_{rs} same as given in Proposition 2.2.

$$\text{Suppose } d_G(u_r, u_s) = 0. \text{ Then } P_{ij} = \begin{cases} 1; & if\ \ d_H(v_i, v_j) = 2 \\ 0; & if\ \ d_H(v_i, v_j) \neq 2 \end{cases}$$

$$\text{Suppose } d_G(u_r, u_s) = 2. \text{ Then } P_{ij} = \begin{cases} 1; & if\ \ i = j \\ 0; & if\ \ i \neq j \end{cases}$$

Also, suppose $d_G(u_r, u_s) \neq 0$ or 2. Then $A_{r,s}$ is a zero matrix of order $m \times m$. Thus,

$$A_{rs} = \begin{cases} A_2(H); & if\ d_G(u_r, u_s) = 0 \\ I_m; & if\ d_G(u_r, u_s) = 2 \\ 0_{m\times m}; & \text{otherwise.} \end{cases}$$

Next, $A_2(G) = [b_{ij}]_{n\times n}$ and $A_2(H) = [c_{ij}]_{m\times m}$. Then

$$A_2(G) \otimes I_m = \begin{bmatrix} b_{11}I_m & b_{12}I_m & \cdots & b_{1n}I_m \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1}I_m & b_{n2}I_m & \cdots & b_{nn}I_m \end{bmatrix}$$

So, $(ij)^{th}$block of$A_2(G) \otimes I_m = [b_{ij} I_m]$ and

$$[b_{ij} I_m] \quad = \quad \begin{cases} I_m; & \text{if } b_{ij} = 1, i.e., d_G(u_i, u_j) = 2 \\ 0_{m\times m}; & \text{otherwise.} \end{cases}$$

In particular $b_{ii} I_m = 0_{m\times m}$.

$$I_n \otimes A_2(H) = \begin{bmatrix} A_2(H) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_2(H) \end{bmatrix}$$

Therefore $(ij)^{th}$ block of $I_n \otimes A_2(H)$ is as follows:

$$I_n \otimes A_2(H) = \quad \begin{cases} A_2(H); & \text{if i = j} \\ 0_{m\times m}; & \text{otherwise.} \end{cases}$$

Suppose $(ij)^{th}$ block of $[A_2(G) \otimes I_m] \oplus [I_n \otimes A_2(H)] = B$. Then

$$B \quad = \quad \begin{cases} A_2(H); & i = j; \text{ i.e., } d_G(u_i, u_j) = 0 \\ I_m; & i \neq j \text{ with } d_G(u_i, u_j) = 2 \\ 0_{m\times m}; & \text{otherwise.} \end{cases}$$

Therefore $(ij)^{th}$ block of $[A_2(G) \otimes I_m] + [I_n \otimes A_2(H)] =$ $(ij)^{th}$ block of $A(G \times_2 H)$. Thus, $[A_2(G) \otimes I_m] + [I_n \otimes A_2(H)] = A(G \times_2 H)$.

3 Spectrum of G ✠ H

In this section, first we recall the spectrum of the graph G. We discuss the spectrum of product graphs in terms of spectrum of factor graphs.

Definition 3.1 [5] For a matrix $A \in M_{n\times n}(\mathbb{R})$, a number λ is an eigenvalue if for some vector $X \neq 0$, $AX = \lambda X$. The vector X is called an eigenvector corresponding to λ. The set of all eigenvalues is the Spectrum of A, and it is denoted by $Spec(A)$, i.e.,
$Spec(A) =$ $\{\lambda \in \mathbb{C} : |\lambda I - A| = 0\}$.

Note that the eigenvalue of G is the eigenvalue of its adjacency matrix $A(G)$ and the spectrum of G is denoted by $Spec(A(G))$.

The following result is known in the usual tensor product $A \otimes B$ of matrices A and B as well as in usual tensor product $G \otimes H$ of graphs G and H.

Proposition 3.2 [9]

(i) Let A and B be two square matrices. $Spec(A \otimes B) = \{\lambda\, \mu : \lambda \in Spec(A),\ \mu \in Spec(B)\}$.

(ii) If G and H are two graphs with n and m vertices respectively, then, $Spec(A(G \otimes H)) = \{\lambda\mu : \lambda \in Spec(A(G)), \mu \in Spec(A(H))\}$.

Using Proposition 3.2(i), we get result similar to (ii) for $G \otimes_2 H$.

Proposition 3.3 Let G and H be two graphs with n and m vertices respectively. Then,

$$Spec(A(G \otimes_2 H)) = \{\lambda\, \mu : \lambda \in Spec(A_2(G)),\ \mu \in Spec(A_2(H))\}.$$

Proof: Let $\lambda \in Spec(A(G))$ and $\mu \in Spec(A(H))$. Let $X = [x_1, x_2, \ldots, x_n]^T$ and $Y = [y_1, y_2, \ldots, y_m]^T$ be eigenvectors corresponding to eigenvalues λ and μ of $A_2(G)$ and $A_2(H)$ respectively. So, $A_2(G)X = \lambda X$ and $A_2(H)Y = \mu Y$. Using kronecker product, we have $(A_2(G)X) \otimes (A_2(H)Y) = (\lambda X) \otimes (\mu Y)$.

Therefore $(A_2(G) \otimes (A_2(H))(X \otimes Y) = \lambda\, \mu(X \otimes Y)$.
By Proposition 2.2, we get $A(G \otimes_2 H)(X \otimes Y) = \lambda\, \mu(X \otimes Y)$.

Thus $\lambda\, \mu$ is an eigenvalue with $X \otimes Y$ as an eigenvector of $A(G \otimes_2 H)$.

The following result is known in the usual cartesian product $G \times H$ of graphs.

Proposition 3.4 [9]
(i) Let A and B be two matrices. $Spec(A \dagger B) = \{\lambda\ + \mu : \lambda \in Spec(A),\ \mu \in Spec(B)\}$.

(ii) If G and H are two graphs with n and m vertices respectively, then,
$$Spec(A(G \times H)) = \{\lambda + \mu : \lambda \in Spec(A(G)),\ \mu \in Spec(A(H))\}.$$

Using Proposition 3.4 (i), we get result similar to (ii) for $G \times_2 H$.

Proposition 3.5 Let G and G be two graphs with n and m vertices respectively. Then,
$$Spec(A(G \times_2 H)) = Spec(A_2(G))\ +\ Spec(A_2(H)).$$
Proof: Let $\lambda \in Spec(A_2(G))$ and $\mu \in Spec(A_2(H))$. Let $X = [x_1, x_2, \ldots, x_n]^T$ and $Y = [y_1, y_2, \ldots, y_m]^T$ be eigenvectors corresponding to eigenvalues λ and μ of $A_2(G)$ and $A_2(H)$ respectively. So, $A_2(G)X = \lambda X$ and $A_2(H)Y = \mu Y$. Using kronecker sum, by Proposition 2.3, we have $A_2(G) \dagger A_2(H) = [A_2(G) \otimes I_m] + [I_n \otimes A_2(H)] = A(G \times_2 H)$. Therefore
$$\begin{aligned} A(G \times_2 H)\ (X \otimes Y) &= \{[A_2(G) \otimes I_m] + [I_n \otimes A_2(H)]\}\ (X \otimes Y) \\ &= [A_2(G)\ X \otimes I_m\ Y] + [I_n\ X \otimes A_2(H)\ Y] \\ &= [\lambda\ X \otimes I_m\ Y] + [I_n\ X \otimes \mu\ Y] \\ &= (\lambda + \mu)(X \otimes\ Y). \end{aligned}$$
Thus $\lambda + \mu$ is an eigenvalue with $X \otimes Y$ as an eigenvector of $A(G \otimes_2 H)$.

4 spectrum of $G \maltese H$ in terms of $A(G)$ and $A(H)$

In this section, we find the relation between $A_2(G)$ and $A^2(G)$, where $A^2(G)$ is $A(G)A(G)$, usual matrix multiplication of $A(G)$ with $A(G)$.

The degree diagonal matrix $D(G)$ is defined as follows:
Definition 4.12 [9] Let $G = (V, E)$ be a graph with vertex set $\{u_1, u_2, \ldots, u_n\}$. Then the degree diagonal matrix $D(G) = [d_{ij}]_{n \times n}$ is defined as

$$d_{ij} \quad = deg(u_i)\delta_{ij} = \quad \begin{cases} deg(u_i); & if\ i = j \\ 0_{m \times m}; & \text{otherwise.} \end{cases}$$

Proposition 4.2 Let G be a simple connected, triangle free and square free graph. Then

$$A_2(G) = A^2(G) - D(G)$$

Proof: Let G be a graph with vertex set $\{u_1, u_2, \ldots, u_n\}$. Now, $A^2(G) = [c_{ij}]$, where $c_{ij} = \sum_{k=1}^{n} a_{ik}\ a_{kj}; 1 \le i, j \le n$.

Suppose $i \neq j$. If $c_{ij} \neq 0$, then for some values of k, a_{ik} and a_{kj} both are non zero, i.e., u_i is adjacent to u_k and u_k is adjacent to u_j in G.

Next, if $u_i \leftrightarrow u_k$ and $u_k \leftrightarrow u_j$, then u_i can not be adjacent with u_j in G, as G is a triangle free graph. So, $u_i \to u_k \to u_j$ gives $d_G(u_i, u_j) = 2$.

Since G is a square free graph, there is a unique path between u_i and u_j of length 2. So, for at most one k, a_{ik} and a_{kj} both are non zero. Thus $c_{ij} = 1$, if $d_G(u_i, u_j) = 2$ and $c_{ij} = 0$, otherwise. Also, it is known that $c_{ii} = deg(u_i) = d_{ii}$. Thus, $A_2(G) = A^2(G) - D(G)$.

Note that $A_2(G)$ is a real symmetric binary matrix.

Remarks 4.3

(i) Let G be connected but not a triangular free graph. Suppose $u_i \to u_k \to u_j \to u_i$ is a triangle in G, Then $d_G(u_i, u_k) = 1 = d_G(u_k, u_j)$. So, $(ij)^{th}$ entry of $A^2(G)$ is 1 but $(ij)^{th}$ entry of $A_2(G)$ is 0, as $d_G(u_i, u_j) \neq 2$. Therefore $A_2(G) \neq A^2(G) - D(G)$.

(ii) If G is connected but not a square free graph, then there may be more than one path between two vertices, say u_i and u_j of length two. Then $(ij)^{th}$ entry of $A^2(G)$ is 2 but $(ij)^{th}$ entry of $A_2(G)$ is 1. So, $A_2(G) \neq A^2(G) - D(G)$.

Corollary 4.4 If G ia a $k-$ regular graph, triangular free and square free graph with n vertices, then $A_2(G) = A^2(G) - kI_n$. Consequently, $Spec(A_2(G)) = \{\lambda^2 - k : \lambda \in Spec(A(G))\}$.
For example, if $G = C_n$, $n \geq 5$, then $A_2(C_n) = A^2(C_n) - 2I_n$.

In next discussion we fix both the graphs G and H connected, triangular free and square free graphs with n and m vertices respectively.

Proposition 4.5 Let G and H be connected graphs.

(i) $A(G \otimes_2 H) = \left[A^2(G) \otimes A^2(H)\right] - \left[A^2(G) \otimes D(H)\right] - \left[D(G) \otimes A^2(H)\right] + [D(G) \otimes D(H)]$.

(ii) $A(G \times_2 H) = \left[A^2(G) \otimes I_m\right] - [D(G) \otimes I_m] + \left[I_n \otimes A^2(H)\right] - [I_n \otimes D(H)]$.

Proof: By Proposition 4.2, $A_2(G) = A^2(G) - D(G)$ and $A_2(H) = A^2(H) - D(H)$.

(i) By Proposition 2.2, we have $A(G \otimes_2 H) = A_2(G) \otimes A_2(H)$. So,
$$\begin{aligned} A(G \otimes_2 H) &= A_2(G) \otimes A_2(H) = [A^2(G) - D(G)] \otimes [A^2(H) - D(H)] \\ &= \left[A^2(G) \otimes A^2(H)\right] - \left[A^2(G) \otimes D(H)\right] - \left[D(G) \otimes A^2(H)\right] + [D(G) \otimes D(H)]. \end{aligned}$$

(ii) By Proposition 2.3, $A(G \times_2 H) = [A_2(G) \otimes I_m] + [I_n \otimes A_2(H)]$. Then by similar arguments as given in case (i), we get the result.

Finally, we obtained the spectrum of $A(G \maltese H)$ in terms of $SpecA(G)$ and $SpecA(H)$. The following result is known in matrix theory.

Theorem 4.6 Let G be $k-$regular and H be $s-$regular connected graphs with n and m vertices respectively. Then
$Spec(A(G \otimes_2 H)) = \{(\lambda^2 - k)\ (\mu^2 - s) : \lambda \in Spec(A(G))\text{and } \mu \in Spec(A(H))\}$.
Proof: Let G and H be regular graphs of n, m vertices with regularity k and s respectively. In addition let X be an eigenvector corresponding to eigenvalue λ of G and Y be an eigenvector corresponding to eigenvalue μ of H. So, $A(G)X = \lambda X$ and $A(H) = \mu Y$. Also $D(G) = kI_n$ and $D(H) = sI_m$.

From Proposition 4.5, we have $A(G \otimes_2 H)\ (X \otimes Y)$
$= \left[A^2(G) \otimes A^2(H)\right](X \otimes Y) - \left[A^2(G) \otimes D(H)\right](X \otimes Y) - \left[D(G) \otimes A^2(H)\right](X \otimes Y)$
$\quad + [D(G) \otimes D(H)]\,(X \otimes Y)$
$= \left[A^2(G)X \otimes A^2(H)Y\right] - \left[A^2(G)X \otimes D(H)Y\right] - \left[D(G)X \otimes A^2(H)Y\right] + [D(G)X \otimes D(H)Y]$
$= \left[\lambda^2 X \otimes \mu^2 Y\right] - \left[\lambda^2 X \otimes sI_m Y\right] - \left[kI_n X \otimes \mu^2 Y\right] + [kI_n X \otimes sI_m Y]$
$= \left[\lambda^2 X \otimes \mu^2 Y\right] - \left[\lambda^2\ X \otimes sY\right] - \left[kX \otimes \mu^2 Y\right] + [kX \otimes s\ Y]$
$= \left[\lambda^2 \mu^2\right](X \otimes Y) - \left[\lambda^2\ s\right](X \otimes Y) - \left[k\mu^2\right](X \otimes Y) + [ks]\ (X \otimes Y)$
$= \left[\lambda^2 \mu^2 - \ \lambda^2\ s - k\mu^2 + k\ s\right](X \otimes Y)$
$= \left[(\lambda^2 - k)\ (\mu^2 - s)\right](X \otimes Y).$
Thus, $Spec(A(G \otimes_2 H)) = \{(\lambda^2 - k)(\mu^2 - s) : \lambda \in Spec(A(G)) \text{ and } \mu \in Spec(A(H))\}$.

Proposition 4.7 Let G and H be $k-$regular and H be $s-$regular connected graphs of n, m vertices respectively. Then

$$Spec(A(G \times_2 H)) = \{(\lambda^2 - k) + (\mu^2 - s) : \lambda \in Spec(A(G)), \mu \in Spec(A(H))\}.$$

Proof: We continue the notations of Theorem 4.8. Also using Proposition 4.6, we have

$$\begin{aligned} A(G \times_2 H)\ (X \otimes Y) &= \{[A^2(G) \otimes I_m] + [I_n \otimes A^2(H)] - [D(G) \otimes I_m] - [I_n \otimes D(H)]\}\ (X \otimes Y) \\ &= [A^2(G)\ X \otimes I_m\ Y] + [I_n\ X \otimes A^2(H)\ Y] - [D(G)X \otimes I_m Y] - [I_n X \otimes D(H)Y] \\ &= [\lambda^2\ X \otimes I_m\ Y] + [I_n\ X \otimes \mu^2\ Y] - [kX \otimes Y] - [X \otimes sY] \\ &= ((\lambda^2 - k) + (\mu^2 - s))(X \otimes\ Y). \end{aligned}$$

Thus, $Spec(A(G \times_2 H)) = \{(\lambda^2 - k) + (\mu^2 - s) : \lambda \in Spec(A(G)), \mu \in Spec(A(H))\}$.

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