# An Iterative Process for a Finite Family of Bregman Totally Quasi-Asymptotically Nonexpansive Mappings and a System of Generalized Mixed Equilibrium Problems in Reflexive Banach Spaces

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## Abstract

The equilibrium problem and its generalizations had a great influence in the development of some branches of pure and applied sciences. The equilibrium problems theory provides a natural and novel approach for some problems arising in nonlinear analysis, physics and engineering, image reconstruction, economics, finance, game theory and optimization. In recent times, there were many methods in order to solve the equilibrium problem and its generalizations. Some authors proposed many iterative methods and studied the convergence of such iterative methods for equilibrium problems and nonexpansive mappings in the setting of Hilbert spaces and Banach spaces. Note that a generalized mixed equilibrium problem is a generalization of an equilibrium problem and a Bregman totally quasi-asymptotically nonexpansive mapping is a generalization of a nonexpansive mapping in reflexive Banach spaces. The purpose of this paper is to combine the parallel method with the Bregman distance and the Bregman projection in order to introduce a new parallel hybrid iterative process which is to find common solutions of a finite family of Bregman totally quasi-asymptotically nonexpansive mappings and a system of generalized mixed equilibrium problems. After that, we prove that the proposed iteration strongly converges to the Bregman projection of initial element on the intersection of common fixed point set of a finite family of Bregman totally quasi-asymptotically nonexpansive mappings and the solution set of a system of generalized mixed equilibrium problems in reflexive Banach spaces. As application, we obatin some strong convergence results for a Bregman totally quasi-asymptotically nonexpansive mapping and a generalized mixed

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equilibrium problem in reflexive Banach spaces. These results are extensions and improvements to the main results in [7, 8]. In addition, a numerical example is provided to illustrate for the obtained result.

*Key words:* Bregman totally quasi-asymptotically nonexpansive mapping, iterative method, generalized mixed equilibrium problem, reflexive Banach space, strong convergence.

# 1　Introduction

In 2008, Peng and Yao [12] generalized the equilibrium problem to the generalized mixed equilibrium problem. Assume that $W$ is a real Banach space, $\Omega$ is a nonempty, closed and convex subset of $W$, $W^*$ is a the dual space of $W$. Let $f : \Omega \times \Omega \to \mathbb{R}$, $\varphi : \Omega \to \mathbb{R}$ be two functions and $\psi : \Omega \to W^*$ be a mapping. We denote the value of $u^* \in W^*$ at $u \in X$ by $\langle u^*, u \rangle$. The generalized mixed equilibrium problem (GMEP) is to find $u \in \Omega$ such that

$f(u,v) + \langle \psi(u), v - u \rangle + \varphi(v) \geq \varphi(u), \forall v \in \Omega.$

The set of solutions of (GMEP) is denoted by

$GMEP(f, \varphi, \psi) = \{u \in \Omega : f(u,v) + \langle \psi(u), v - u \rangle + \varphi(v) \geq \varphi(u), \forall v \in \Omega\}.$

When $\varphi \equiv 0$ and $\psi \equiv 0$, the problem (GMEP) is reduced to the equilibrium problem (EP) which is to find $u \in \Omega$ such that $f(u,v) \geq 0$ for all $v \in \Omega$. In recent times, some authors constructed many iterative methods for finding common solutions of fixed points set of nonlinear mappings and solution set of (EP) or (GMEP) in reflexive Banach spaces with the Bregman distance [6, 10, 14, 16]. In 2018, Kazmi et al. [8] extended and improved the main results in [7] from an equilibrium problem and a fixed point problem for an asymptotically quasi-$\phi$-nonexpansive mapping in intermediate sense in uniformly smooth and strictly convex Banach spaces to a generalized equilibrium and fixed point problem for a Bregman relatively nonexpansive mapping in reflexive Banach spaces.

The notion of a Bregman totally quasi-asymptotically nonexpansive mapping is introduced by Chang et al. [5] in 2014. This mapping is a generalization of many mappings which relate to the Bregman distance in reflexive Banach spaces. An interesting work naturally raised is to extend and generalize the existing convergence results to (GMEP) and Bregman totally quasi-asymptotically nonexpansive mappings in reflexive Banach spaces. In this paper, motivated by the iteration processes in [7, 8], we introduce a new hybrid iterative method for a finite family of Bregman totally quasi-asymptotically nonexpansive mappings and a system of generalized mixed equilibrium problems in reflexive Banach spaces. By using some suitable conditions, we prove that the proposed iteration strongly converges to the Bregman projection of initial element on the intersection of common fixed point set of a finite family of Bregman totally quasi-asymptotically nonexpansive mappings and the solution set of a system of generalized mixed equilibrium problems in reflexive Banach spaces. In addition, a numerical example is given to illustrate for the obtained results.

# 2　Preliminaries

Throughout this paper, we assume that $W$ is a real reflexive Banach space, $\Omega$ is a nonempty, closed and convex subset of $W$, $W^*$ is a the dual space of $W$, $g : W \to (-\infty, +\infty]$ is a lower semi

continuous, convex and proper function, $\mathrm{dom}\,g = \{u \in W : g(u) < +\infty\}$ is the domain of $g$. For any $u \in \mathrm{int}(\mathrm{dom}\,g),\ v \in W,$ we denote by

$$g'(u,v) = \lim_{\lambda \to 0^+} \frac{g(u + \lambda v) - g(u)}{\lambda} \qquad (1.1)$$

the right-hand derivative of $g$ at $u$ in the direction $v$. The function $g$ is called *Gâteaux differentiable at* $u$ if the limit (1.1) exists for all $v$. Then the gradient of $g$ at $u$ is $\nabla g(u),$ which is defined by $\langle \nabla g(u), v \rangle = g'(u,v)$ for all $v \in W.$ The function $g$ is called *Fréchet differentiable at* $u$ if the limit (1.1) is attained uniformly in $\| v \| = 1$. The function $g$ is called be *uniformly Fréchet differentiable on a subset* $\Omega$ of $W$ if the limit (1.1) is attained uniformly for $u \in \Omega$ and $\| v \| = 1$.

**Remark 2.1.** (1) ([1], Theorem 1.8) If $g$ is uniformly Fréchet differentiable, then $g$ is uniformly continuous.

(2) ([1], Proposition 1.1.11). If $g$ is Gâteaux differentiable, then $g$ is bounded on bounded sets if and only if $\nabla g$ is bounded on bounded sets.

(3) ([14], Proposition 1). If $g$ is uniformly Fréchet differentiable and bounded on bounded subsets, then $\nabla g$ is uniformly continuous on bounded subsets of $W^*$.

Let $u \in \mathrm{int}(\mathrm{dom}\,g),$ the Fenchel conjugate of $g$ is the function $g^* : W^* \to (-\infty, +\infty]$ defined by

$g^*(u^*) = \sup\{\langle u^*, u \rangle - g(u) : u \in W\}, \forall u^* \in W^*.$

**Definition 2.2** ([5], Definition 2.2). The function $g : W \to (-\infty, +\infty]$ is called *Legendre* if

(L1) $\mathrm{int}(\mathrm{dom}\,g) \neq \varnothing,\ g$ is Gâteaux differentiable on $\mathrm{int}(\mathrm{dom}\,g)$ and $\mathrm{dom}(\nabla g) = \mathrm{int}(\mathrm{dom}\,g).$

(L2) $\mathrm{int}(\mathrm{dom}\,g^*) \neq \varnothing,\ g^*$ is Gâteaux differentiable on $\mathrm{int}(\mathrm{dom}\,g^*)$ and $\mathrm{dom}(\nabla g^*) = \mathrm{int}(\mathrm{dom}\,g^*).$

**Remark 2.3** ([5], Remark 2.3). *Assume that* $g : W \to (-\infty, +\infty]$ *is a Legendre function. Then*

*(1)* $g$ *is Legendre if and only if* $g^*$ *is Legendre.*

*(2)* $\nabla g = (\nabla g^*)^{-1},\ \mathrm{ran}(\nabla g) = \mathrm{dom}(\nabla g^*)$ *and*

$\mathrm{ran}(\nabla g^*) = \mathrm{dom}(\nabla g) = \mathrm{int}(\mathrm{dom}\,g),$ *where* $\mathrm{ran}(\nabla g)$ *is the range of* $\nabla g$ *.*

**Definition 2.4** ([4], p.324). Let $g : W \to (-\infty, +\infty]$ be Gâteaux differentiable. Then, the function $D_g : \mathrm{dom}\,g \times \mathrm{int}(\mathrm{dom}\,g) \to [0, \infty),$ defined by $D_g(u,v) = g(u) - g(v) - \langle \nabla g(v), u - v \rangle$ is called the *Bregman distance* with respect to $g.$

From the definition, for all $u \in \mathrm{dom}\,g$ and $v, w \in \mathrm{int}(\mathrm{dom}\,g),$ we have

$D_g(u,v) + D_g(v,w) - D_g(u,w) = \langle \nabla g(w) - \nabla g(v), u - v \rangle.$

Let $g : W \to (-\infty, +\infty]$ be Gâteaux differentiable and $V_g : W \times W^* \to [0, \infty)$ be defined by

$$V_g(u, u^*) = g(u) - \langle u^*, u \rangle + g^*(u^*) \text{ for all } u \in W \text{ and } u^* \in W^*.$$

**Remark 2.5** *Let* $g : W \to (-\infty, +\infty]$ *be a Gâteaux differentiable function. Then*

(1) ([9], Lemma 3.2) *For any* $u \in W$ *and* $u^* \in W^*,$ *we have* $V_g(u, u^*) = D_g(u, \nabla g^*(u^*)).$

(2) ([10], p.7]) $V_g$ *is convex in the second variable. In addition, for any* $u \in \mathrm{dom}\,g,$

$\{u_k\}_{k=1}^m \subset \mathrm{int}(\mathrm{dom}\,g)$ *and* $\{t_k\}_{k=1}^m \subset [0,1]$ *with* $\sum_{k=1}^m t_k = 1,$ *we have* $D_g(u, \nabla g^*(\sum_{k=1}^m t_k \nabla g(u_k))) \leq \sum_{k=1}^m t_k D_g(u, u_k).$

**Definition 2.6** ([2], p.69)**.** Let $g : W \to (-\infty, +\infty]$ be Legendre and $\Omega$ be a nonempty, convex and closed subset of $\text{int}(\text{dom}g)$. The *Bregman projection* of $u \in \text{int}(\text{dom}g)$ onto $\Omega$ is the unique vector $P_\Omega^g(u) \in \Omega$ statisfying $D_g(P_\Omega^g(u), u) = \inf\{D_g(v, u) : v \in \Omega\}$.

**Remark 2.7** ([8], Remark 2.1)**.** (1) *If $W$ is a uniformly smooth, strictly convex Banach space and* $g(u) = \| u \|^2$ *for all* $u \in W$, *then* $\nabla g(u) = 2Ju$ *for all* $u \in W$ *and* $J$ *is the normalized duality mapping which is defined by* $Ju = \{u^* \in W^* : \langle u, u^* \rangle = \| u \|^2 = \| u^* \|^2\}$ *for all* $u \in W$. *Then the Bregman distance* $D_g(u, v)$ *is reduced into the Lyapunov function* $\phi(u, v)$ *which is defined by* $\phi(u, v) = \| u \|^2 - 2\langle u, Jv \rangle + \| v \|^2$ *for all* $u, v \in W$. *Therefore, the Bregman projection* $P_\Omega^g(u)$ *is reduced into the generalized projection* $\prod_\Omega(u)$ *in a smooth space which is defined by* $\phi(\prod_\Omega(u), u) = \min\{\phi(v, u) : v \in \Omega\}$.

(2) *If $W$ is a Hilbert space and $g(u) = \| u \|^2$ for all $u \in W$, then $D_g(u, v) = \| u - v \|^2$ for all $u, v \in W$ and $J$ is the identity mapping. Therefore, the Bregman projection $P_\Omega^g(u)$ is reduced into the metric projection from $W$ onto $\Omega$.*

**Definition 2.8** ([15], p.1)**.** Let $g : W \to (-\infty, +\infty]$ be Gâteaux differentiable. Then

(1) $g$ is called *totally convex at* $u \in \text{int}(\text{dom}g)$ if any $\varepsilon > 0$, we have

$v_g(u, \varepsilon) = \inf\{D_g(v, u) : y \in \text{dom}\, g, \| v - u \| = \varepsilon\} > 0.$

(2) $g$ is called *totally convex* if $g$ is totally convex at every point $u \in \text{int}(\text{dom}f)$.

(3) $g$ is called *totally convex on bounded subsets of $W$* if any nonempty bounded subset $E$ of $W$ and $t > 0$, we have $v_g(E, \varepsilon) = \inf\{v_g(u, \varepsilon) : u \in E \cap \text{dom}\, g\} > 0.$

**Proposition 2.9** ([15], Proposition 2.2)**.** *Let $g : W \to (-\infty, +\infty]$ be Gâteaux differentiable. Then $g$ is totally convex at $u \in \text{int}(\text{dom}g)$ if and only if for any sequense $\{v_n\} \subset \text{int}(\text{dom}g)$ such that* $\lim_{n \to \infty} D_g(v_n, u) = 0$, *we have* $\lim_{n \to \infty} \| v_n - u \| = 0.$

**Proposition 2.10** ([2], Lemma 2.1.2)**.** *Let $g : W \to (-\infty, +\infty]$ be Gâteaux differentiable. Then $g$ is totally convex on bounded sets if and only if for any sequence $\{u_n\} \subset \text{dom}g$ and $\{v_n\} \subset \text{int}(\text{dom}g)$ such that $\{u_n\}$ is bounded and* $\lim_{n \to \infty} D_g(v_n, u_n) = 0$, *we have* $\lim_{n \to \infty} \| v_n - u_n \| = 0.$

**Proposition 2.11** ([3], Corollary 4.4)**.** *Let $g : W \to (-\infty, +\infty]$ be Gâteaux differentiable and totally convex on* $\text{int}(\text{dom}g)$, $\Omega$ *be a nonempty, closed and convex subset and* $u \in \text{int}(\text{dom}g)$. *Then*

*(1) $w = P_\Omega^g(u)$. if and only if $\langle \nabla g(u) - \nabla g(w), w - v \rangle \geq 0, v \in \Omega$*

*(2) $D_g(v, P_\Omega^g(u)) + D_g(P_\Omega^g(u), u) \leq D_g(v, u), v \in \Omega.$*

**Proposition 2.12** ([13] Lemma 1)**.** *If $g : W \to \mathbb{R}$ is Gâteaux differentiable and totally convex on $W$, $u \in W$ and $\{u_n\} \subset W$ satisfying $\{D_g(u_n, u)\}$ is bounded, then the sequence $\{u_n\}$ is bounded.*

**Proposition 2.13** ([16], Proposition 2.3)**.** *If $g$ is Legendre such that $\nabla g^*$ is bounded on bounded subsets, $u \in W$ and $\{u_n\} \subset W$ satisfying $\{D_g(u, u_n)\}$ is bounded, then the sequence $\{u_n\}$ is bounded.*

**Definition 2.14** ([17], p.203, p.207, p.221)**.** Let $S_1 = \{u \in W : \|u\| < 1\}$ and $B_\varepsilon = \{u \in W : \|u\| \leq \varepsilon\}$ for some $\varepsilon > 0$. Then

(1) $g : W \to \mathbb{R}$ is called *uniformly convex on bounded subsets* if $\rho_\varepsilon(\lambda) > 0$ for all $\lambda, \varepsilon > 0$, where

$\rho_\varepsilon : [0, \infty) \to [0, \infty)$ is defined by $\rho_\varepsilon(\lambda) = \inf\limits_{u,v \in B_\varepsilon, \|u-v\| = \lambda, \delta \notin (0,1)} \dfrac{\delta g(u) + (1-\delta) g(v) - g(\delta u + (1-\delta) v)}{\delta(1-\delta)}.$

(2) $g : W \to \mathbb{R}$ is called *uniformly smooth on bounded subsets* if $\lim\limits_{\lambda \to 0} \dfrac{\sigma_\varepsilon(\lambda)}{\lambda} = 0$ for all $\varepsilon > 0$, where

$\sigma_\varepsilon : [0, \infty) \to [0, \infty)$ is defined by $\sigma_\varepsilon(\lambda) = \sup\limits_{u \in B_\varepsilon, v \in S_1, \delta \in (0,1)} \dfrac{\delta g(u + (1-\delta)\lambda v) + (1-\delta)g(u - \delta\lambda v) - g(u)}{\delta(1-\delta)}.$

**Remark 2.15** ([11], p.7)**.** *The function $g$ is uniformly convex on bounded subsets if and only if $g$ is totally convex on bounded subsets.*

**Definition 2.16** ([9], p.509)**.** The function $g : W \to (-\infty, +\infty]$ is called *strongly coercive* if

$\lim\limits_{\|u\| \to +\infty} \dfrac{g(u)}{\|u\|} = +\infty.$

**Proposition 2.17** ([17], Proposition 3.6.3)**.** *Let $g : W \to \mathbb{R}$ be strongly coercive, continuous and convex. Then $g$ is bounded on bounded subsets and uniformly smooth on bounded subsets if and only if $\mathrm{dom}(g^*) = W^*$, $g^*$ is strongly coercive and uniformly convex on bounded subsets.*

**Proposition 2.18** ([17], Proposition 3.6.4)**.** *Let $g : W \to \mathbb{R}$ be convex, continuous and bounded on bounded subsets of $W$. Then the following statements are equivalent.*

*(1) $g$ is uniformly convex on bounded subsets and strongly coercive.*

*(2) $\mathrm{Dom}(g^*) = W^*$, $g^*$ is bounded and uniformly smooth on bounded subsets.*

*(3) $\mathrm{Dom}(g^*) = W^*$, $g^*$ is Fréchet differentiable and $\nabla g^*$ is uniformly continuous on bounded subsets.*

**Lemma 2.19** ([11], Lemma 2.2)**.** *Let $r > 0$ and $g : W \to \mathbb{R}$ be convex and uniformly convex on bounded subsets. Then $g(\sum\limits_{n=1}^{m} a_n u_n) \leq \sum\limits_{n=1}^{m} a g(u_n) - a_i a_j \rho_\varepsilon(\|u_i - u_j\|)$ with $i, j \in \{1, 2, ..., m\}$, $u_n \in B_\varepsilon = \{u \in W : \|u\| \leq \varepsilon\}$ and $a_n \in [0,1]$ such that $\sum\limits_{n=1}^{m} a_n = 1$, and the function $\rho_\varepsilon$ is defined as in Definition 2.13.*

We denote by $F(S) = \{w \in \Omega : Sw = w\}$ the set of fixed points of $S : \Omega \to \Omega$.

**Definition 2.20** ([5], Definition 2.10)**.** Let $S : \Omega \to \Omega$ be a mapping, $D_g$ be the Bregman distance, $\phi$ be the Lyapunov function. Then

(1) $S$ is called a *Bregman quasi-asymptotically nonexpansive mapping* if $F(S) \neq \varnothing$ and there exists a real sequence $\{\delta_n\} \subset [1, \infty)$ with $\lim\limits_{n \to \infty} \delta_n = 1$ such that $D_g(u, S^n v) \leq \delta_n D_g(u, v)$ for all $v \in \Omega$ and $u \in F(S)$.

(2) $S$ is called a *Bregman totally quasi-asymptotically nonexpansive mapping* if $F(S) \neq \varnothing$ and there exist nonnegative real sequences $\{\alpha_n\}, \{\beta_n\}$ with $\lim\limits_{n \to \infty} \alpha_n = \lim\limits_{n \to \infty} \beta_n = 0$ and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\zeta(0) = 0$ such that for all $v \in \Omega$ and $u \in F(S)$, we have $D_g(u, S^n v) \leq D_g(u, v) + \alpha_n \zeta(D_g(u, v)) + \beta_n$.

(3) $S$ is called a *Bregman firmly nonexpansive mapping* if for all $u,v \in \Omega$, we have

$\langle \nabla g(Su) - \nabla g(Sv), Su - Sv \rangle \leq \langle \nabla g(u) - \nabla g(v), Su - Sv \rangle$. (4) $S$ is called a *Bregman quasi-nonexpansive mapping* if $F(S) \neq \varnothing$ and for all $u \in \Omega$ and $v \in F(S)$, we have $D_g(u, Sv) \leq D_g(u,v)$.

(5) $S$ is called a *totally quasi-$\phi$-asymptotically nonexpansive mapping* if $F(S) \neq \varnothing$ and there exist nonnegative real sequences $\{\alpha_n\}, \{\beta_n\}$ with $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$ and a strictly increasing continuous function $\zeta : \mathbb{R}^+ \to \mathbb{R}^+$ with $\zeta(0) = 0$ such that for all $v \in \Omega$ and $u \in F(S)$, we have $\phi(u, S^n v) \leq \phi(u,v) + \alpha_n \zeta(\phi_g(u,v)) + \beta_n$.

(6) $S$ is called a *totally quasi-$\phi$-asymptotically nonexpansive mapping* if $F(S) \neq \varnothing$ and there exists a real sequence $\{\delta_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} \delta_n = 1$ such that $\phi(u, S^n v) \leq \delta_n \phi(u,v)$ for all $v \in \Omega$ and $u \in F(S)$.

**Remark 2.21** ([5], p.42). (*1*) *If $S$ is a Bregman quasi-asymptotically nonexpansive mapping, then $S$ is a Bregman totally quasi-asymptotically nonexpansive mapping with $\zeta(\lambda) = \lambda$ for all $\lambda \geq 0$, $\alpha_n = \delta_n - 1$ with $\delta_n \geq 1$ satisfying $\lim_{n\to\infty} \delta_n = 1$ and $\beta_n = 0$; but the converse is not true.*

*(2) If $S$ is a quasi-$\phi$-asymptotically nonexpansive mapping, then $S$ is a totally quasi-$\phi$-asymptotically nonexpansive mapping with $\zeta(\lambda) = \lambda$ for all $\lambda \geq 0$, $\alpha_n = \delta_n - 1$ with $\delta_n \geq 1$ satisfying $\lim_{n\to\infty} \delta_n = 1$ and $\beta_n = 0$.*

*(3) If $S$ is a Bregman firmly nonexpansive mapping, then $S$ is a Bregman quasi-nonexpansive mapping.*

**Definition 2.22** ([18], Definition 2.10). Let $S : \Omega \to \Omega$ be a mapping. Then

(1) $S$ is called *closed* if any sequence $\{u_n\}$ in $\Omega$ such that $\lim_{n\to\infty} u_n = u \in \Omega$ and $\lim_{n\to\infty} Su_n = v \in \Omega$, we have $Su = v$.

(2) $S$ is called *uniformly asymptotically regular* on $\Omega$ if for all bounded subset $U$ of $\Omega$ we have $\limsup_{n\to\infty} \sup_{x\in U} \| S^{n+1}u - S^n u \| = 0$.

**Lemma 2.23** ([5], Lemma 2.16). *Suppose that $g : W \to (-\infty, +\infty]$ is a Legendre, totally convex on bounded subsets of $W$, and $S : \Omega \to \Omega$ is a closed and Bregman totally quasi-asymptotically nonexpansive mapping. Then $F(S)$ is convex and closed.*

In order to slove (GMEP), we suppose that $f, \varphi, \psi$ satisfy the following conditions.

(C1) $f(u,u) = 0, \forall u \in \Omega$.

(C2) $f(u,v) + f(v,u) \leq 0, \forall u, v \in \Omega$.

(C3 $\limsup_{\lambda\to 0} f(\lambda w + (1-\lambda)u, v) \leq f(u,v), \forall u, v, w \in \Omega$

(C4) For each $u \in \Omega$, $v \mapsto f(u,v)$ is convex and lower semi-continuous.

(C5) $\varphi : \Omega \to \mathbb{R}$ is convex and lower semi-continuous.

(C6) $\psi : \Omega \to X^*$ is continuous monotone.

**Lemma 2.24** ([6], Lemma 2.8). *Assume that $g : W \to \mathbb{R}$ is a Legendre and strongly coercive function, the functions $f, \varphi, \psi$ satisfy the conditions (C1)-(C6). Consider the mapping $\mathrm{Res}_{f,\varphi,\psi}^g : W \to 2^\Omega$ which is defined by*

$$\mathrm{Res}_{f,\varphi,\psi}^g(u) = \{w \in \Omega : f(w,v) + \varphi(v) + \langle \psi(w), v - w \rangle + \langle \nabla g(w) - \nabla g(u), v - w \rangle \geq \varphi(w), \forall v \in \Omega\}.$$

*Then*

*(1)* $\mathrm{Dom}(\mathrm{Res}^g_{f,\varphi,\psi}) = W$, $\mathrm{Res}^g_{f,\varphi,\psi}$ *is a single-valued and Bregman firmly nonexpansive mapping.*

*(2)* $F(\mathrm{Res}^g_{f,\varphi,\psi}) = GMEP(f,\varphi,\psi)$, *and* $GMEP(f,\varphi,\psi)$ *is convex and closed.*

*(3) For all* $u \in W$ *and* $v \in F(\mathrm{Res}^g_{f,\varphi,\psi})$, *we have* $D_g(v, \mathrm{Res}^g_{f,\varphi,\psi}(u)) + D_g(\mathrm{Res}^g_{f,\varphi,\psi}(u), u) \le D_g(v, u)$.

# 3 Main results

Let $W$ be a real reflexive Banach space and $\Omega$ be a nonempty, closed and convex subset of $W$. We denote by $I = \{1, 2, ..., N\}$ with $N \in \mathbb{N}$. Suppose that for each $i \in I$, $S_i : \Omega \to \Omega$ is Bregman totally quasi-asymptotically nonexpansive mapping with nonnegative real sequences $\{\alpha_n^{(i)}\}$ and $\{\beta_n^{(i)}\}$ satisfying $\lim\limits_{n \to \infty} \alpha_n^{(i)} = \lim\limits_{n \to \infty} \beta_n^{(i)} = 0$ and strictly increasing continuous functions $\zeta^{(i)} : [0, \infty) \to [0, \infty)$ with $\zeta^{(i)}(0) = 0$. Put $\alpha_n = \max\{\alpha_n^{(i)} : i \in I\}$, $\beta_n = \max\{\beta_n^{(i)} : i \in I\}$ and $\zeta(t) = \max\{\zeta^{(i)}(t) : i \in I\}$ for all $t \ge 0$. Then $\lim\limits_{n \to \infty} \alpha_n = \lim\limits_{n \to \infty} \beta_n = 0$, $\zeta(0) = 0$ and $D_g(p, S_i^n u) \le D_g(p, u) + \alpha_n \zeta(D_g(p, u)) + \beta_n$ for all $u \in \Omega$, $p \in \bigcap\limits_{i \in I} F(S_i)$ and for all $i \in I$.

**Theorem 3.1.** *Assume that* $W$ *is a real reflexive Banach space, and* $\Omega$ *is a nonempty, closed and convex subset of* $W$. *Let* $g : W \to \mathbb{R}$ *be Legendre, strongly coercive on* $W$, *and bounded, totally convex, uniformly Fréchet differentiable on bounded subsets of* $W$. *For each* $k \in K := \{1, 2, ..., M\}$ *with* $M \in \mathbb{N}$, *the functions* $f_k, \varphi_k, \psi_k$ *satisfy the conditions (C1)-(C6). For each* $i \in I$, $S_i : \Omega \to \Omega$ *is a closed, uniformly asymptotically regular and Bregman totally quasi-asymptotically nonexpansive mapping with nonnegative real sequences* $\{\alpha_n^{(i)}\}$ *and* $\{\beta_n^{(i)}\}$ *satisfying* $\lim\limits_{n \to \infty} \alpha_n^{(i)} = \lim\limits_{n \to \infty} \beta_n^{(i)} = 0$ *and strictly increasing continuous function* $\zeta^{(i)} : [0, \infty) \to [0, \infty)$ *with* $\zeta^{(i)}(0) = 0$ *such that*

$$\mathcal{F} = (\bigcap_{i \in I} F(S_i)) \cap (\bigcap_{k \in K} GMEP(f_k, \varphi_k, \psi_k))$$

*is nonempty and bounded. Let* $\{u_n\}$ *be a sequence generated by*

$$\begin{cases} u_1, v_1 \in \Omega, \Omega_1 = \Omega \\ w_n^{(i)} = \nabla g^*[a_n \nabla g(v_n) + (1 - a_n)\nabla g(S_i^n u_n)] \\ i_n = \arg\max\{D_g(u_n, w_n^{(i)}) : i \in I\}, \overline{w}_n = w_n^{(i_n)} \\ v_{n+1} = H^M(\overline{w}_n) \\ \Omega_{n+1} = \{u \in \Omega_n : D_g(u, v_{n+1}) \le a_n D_g(u, v_n) + (1 - a_n)D_g(u, u_n) + \xi_n\} \\ u_{n+1} = P^g_{\Omega_{n+1}}(u_1), \forall n \ge 2 \end{cases} \tag{3.1}$$

*where* $\xi_n = (1 - a_n)[\alpha_n \sup\{\zeta(D_g(u, u_n)) : u \in \mathcal{F}\} + \beta_n]$, *the function* $R_k := \mathrm{Res}^g_{f_k, \varphi_k, \psi_k}$ *is defined as in Lemma 2.24,* $H^k = R_k(R_{k-1}(R_{k-2}(...(R_1))))$ *for all* $k \in K$, $\{a_n\} \subset [0, 1]$ *such that* $\lim\limits_{n \to \infty} a_n = 0$. *Then the sequence* $\{u_n\}$ *strongly converges to* $p = P^g_{\mathcal{F}}(u_1)$.

**Proof.** We divide the proof of this theorem into six steps.

**Step 1.** *We claim that* $P^g_{\mathcal{F}}(u_1)$ *is well-defined.* Indeed, by Lemma 2.23 and Lemma 2.24, we find that $F(S_i)$ and $GMEP(f_k, \varphi_k, \psi_k)$ are closed and convex for all $i \in I$ and $k \in K$. Therefore, by combining this

with the assumption, we obtain that $\mathcal{F} = (\bigcap_{i \in I} F(S_i)) \cap (\bigcap_{k \in K} GMEP(f_k, \psi_k, \varphi_k))$ is a nonempty, closed and convex subset of $\Omega$. This fact ensures that $P_{\mathcal{F}}^g(u_1)$ is well-defined.

**Step 2.** *We claim that $P_{\Omega_{n+1}}^g(u_1)$ is well-defined.* We first claim that $\Omega_n$ is convex and closed for all $n \geq 1$ by mathematical induction. Obviously, for $n = 1$, we have $\Omega_1 = \Omega$ is closed and convex. Now we suppose that $\Omega_m$ is convex and closed for some $m \geq 1$. Then, by the definition of $\Omega_{m+1}$, we have

$$\Omega_{m+1} = \{u \in \Omega_m : a_m \langle \nabla g(v_m), u - v_m \rangle + (1 - a_m) \langle \nabla g(u_m), u - u_m \rangle - \langle \nabla g(v_{m+1}), u - v_{m+1} \rangle$$
$$\leq g(v_{m+1}) - a_m g(v_m) - (1 - a_m) g(u_m) + \xi_m \}. \tag{3.2}$$

By combining (3.2) with the continuity of $\nabla g(.)$, we get that $\Omega_{m+1}$ is convex and closed. Therefore, $\Omega_n$ is convex and closed for all $n \geq 1$.

Next, we will prove that $\mathcal{F} \subset \Omega_n$ for all $n \geq 1$ by mathematical induction. Obviously, we have $\mathcal{F} \subset \Omega = \Omega_1$. Now, we suppose that $\mathcal{F} \subset \Omega_m$ for some $m \geq 1$. We show that $\mathcal{F} \subset \Omega_{m+1}$. Indeed, for any $u \in \mathcal{F}$, we get $u \in \Omega_m$. By Remark 2.21.(2) and Lemma 2.24, we find that $H^M$ is a Bregman quasi-nonexpansive mapping. Then

$$D_g(u, v_{m+1}) = D_g(u, H^M(\bar{w}_m)) \leq D_g(u, \bar{w}_m). \tag{3.3}$$

Furthermore, by Remark 2.5.(2) and the fact that $S_i$ is a Bregman totally quasi-asymptotically nonexpansive mapping, we obtain

$$D_g(u, \bar{w}_m) = D_g(u, w_m^{(i_m)})$$
$$= D_g(u, \nabla g^*[a_m \nabla g(v_m) + (1 - a_m) \nabla g(S_i^m u_m)])$$
$$\leq a_m D_g(u, v_m) + (1 - a_m) D_g(u, S_i^m u_m)$$
$$\leq a_m D_g(u, v_m) + (1 - a_m)[D_g(u, u_m) + \alpha_m \zeta(D_g(u, u_m)) + \beta_n]$$
$$= a_m D_g(u, v_m) + (1 - a_m) D_g(u, u_m) + \xi_m. \tag{3.4}$$

It follows from (3.3) and (3.4) that

$$D_g(u, v_{m+1}) \leq a_m D_g(u, v_m) + (1 - a_m) D_g(u, u_m) + \xi_m. \tag{3.5}$$

This leads to $u \in \Omega_{m+1}$ and hence $\mathcal{F} \subset \Omega_{m+1}$. Therefore, $\mathcal{F} \subset \Omega_n$ for all $n \geq 1$. By combining this with $\mathcal{F} \neq \varnothing$, we have $\Omega_n \neq \varnothing$. By the above, we find that $\Omega_{n+1}$ is nonempty, closed and convex. This proves that $P_{\Omega_{n+1}}^g(u_1)$ is well-defined.

**Step 3.** *We claim that $\{u_n\}$ and $\{v_n\}$ are bounded, and the limit $\lim_{n \to \infty} D_g(u_n, u_1)$ exists.* Indeed, it follows $u_n = P_{\Omega_n}^g(u_1)$ and Proposition 2.11 that

$$D_g(y, u_n) + D_g(u_n, u_1) \leq D_g(y, u_1) \tag{3.6}$$

for all $y \in \Omega_n$. Let $u \in \mathcal{F}$. Since $\mathcal{F} \subset \Omega_n$, we get $u \in \Omega_n$. By choosing $y = u$ in (3.6), we obtain

$$D_g(u, u_n) + D_g(u_n, u_1) \leq D_g(u, u_1). \tag{3.7}$$

This implies that $D_g(u_n, u_1) \leq D_g(u, u_1) - D_g(u, u_n) \leq D_g(u, u_1)$. Therefore, $\{D_g(u_n, u_1)\}$ is bounded. Then, from Proposition 2.12, we find that the sequence $\{u_n\}$ is bounded. Furthermore, by (3.7), we

have $D_g(u,u_n) \le D_g(u,u_1) - D_g(u_n,u_1) \le D_g(u,u_1)$. This proves that $\{D_g(u,u_n)\}$ is bounded. By combining this with the boundedness of $\{\alpha_n\}$ and $\{\beta_n\}$, we find that there exists $b > 0$ such that

$$b = \sup_{n \in \mathbb{N}^*} \{\alpha_n \sup\{\zeta(D_g(u,u_n)) : u \in \mathcal{F}\} + \beta_n\}.$$

Put $a = \max\{D_g(u,v_1), \sup_{n \in \mathbb{N}^*} D_g(u,u_n)\}$. We will claim that $D_g(u,v_n) \le a + b$ for all $n \ge 1$ by mathematical induction. Indeed, we have $D_g(u,v_1) \le a \le a + b$. Now, we suppose that $D_g(u,v_m) \le a + b$ for some $m \ge 1$. Then, by (3.5), we have

$$D_g(u,v_{m+1}) \le a_m D_g(u,v_m) + (1-a_m)D_g(u,u_m) + (1-a_m)b$$

$$\le a_m(a+b) + (1-a_m)a + (1-a_m)b = a + b.$$

This imples that $D_g(u,v_n) \le a + b$ for all $n \ge 1$ and hence the sequence $\{D_g(u,v_n)\}$ is bounded. Furthermore, by Proposition 2.18, we find that $g^*$ is bounded on bounded subsets. By Remark 2.1(1), we conclude that $\nabla g^*$ is bounded on bounded subsets. It follows from Proposition 2.13 and the boundedness of $\{D_g(u,v_n)\}$ that $\{v_n\}$ is bounded.

Moreover, we have $u_{n+1} = P_{\Omega_{n+1}}^g(u_1) \in \Omega_{n+1} \subset \Omega_n$. By choosing $y = u_{n+1}$ in (3.6), we get

$$D_g(u_{n+1},u_n) + D_g(u_n,u_1) \le D_g(u_{n+1},u_1).$$

This implies that $D_g(u_n,u_1) \le D_g(u_{n+1},u_1) - D_g(u_{n+1},u_n) \le D_g(u_{n+1},u_1)$. This proves that $\{D_g(u_n,u_1)\}$ is a nondecreasing sequence. By combining this with the boundedness of the sequence $\{D_g(u_n,u_1)\}$, we conclude that the limit $\lim_{n \to \infty} D_g(u_n,u_1)$ exsits.

**Step 4.** *We claim that* $\lim_{n \to \infty} u_n = p \in \Omega$. Indeed, for $m > n$, we get that $u_m = P_{\Omega_m}^g(u_1) \in \Omega_m \subset \Omega_n$. By choosing $y = u_m$ in (3.6), we get $D_g(u_m,u_n) + D_g(u_n,u_1) \le D_g(u_m,u_1)$. This leads to

$$0 \le D_g(u_m,u_n) \le D_g(u_m,u_1) - D_g(u_n,u_1). \tag{3.8}$$

Taking the limit (3.8) as $m,n \to \infty$ and using the existence of $\lim_{n \to \infty} D_g(u_n,u_1)$, we obtain

$$\lim_{m,n \to \infty} D_g(u_m,u_n) = 0. \tag{3.9}$$

It follows from (3.9), the boundedness of the sequence $\{u_n\}$, and Proposition 2.10 we have

$$\lim_{m,n \to \infty} \| u_m - u_n \| = 0. \tag{3.10}$$

This proves that $\{u_n\}$ is a Cauchy sequence in $\Omega$. Since $W$ is a Banach space and $\Omega$ is a closed subset of $W$, there exists $p \in \Omega$ such that $\lim_{n \to \infty} u_n = p$.

**Step 5.** *We claim that* $p \in \mathcal{F}$. First, we prove that $p \in \bigcap_{i \in I} F(S_i)$. Indeed, by choosing $m = n+1$ in (3.9) and (3.10), we obtain

$$\lim_{n \to \infty} D_g(u_{n+1},u_n) = \lim_{n \to \infty} \| u_{n+1} - u_n \| = 0. \tag{3.11}$$

Since $u_{n+1} = P_{\Omega_{n+1}}^g(u_1) \in \Omega_{n+1} \subset \Omega_n$, we get

$$D_g(u_{n+1},v_{n+1}) \le a_n D_g(u_{n+1},v_n) + (1-a_n)D_g(u_{n+1},u_n) + \xi_n. \tag{3.12}$$

Let $u \in \mathcal{F}.$ By the boundedness of $\{D_g(u, u_n)\}$ and $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0,$ we find that $\lim_{n\to\infty} \xi_n = 0.$ By property of the Bregman distance, we have

$$| D_g(u_{n+1}, v_n) | = | \langle \nabla g(v_n) - \nabla g(u_{n+1}), u - u_{n+1} \rangle + D_g(u, v_n) - D_g(u, u_{n+1}) |$$
$$\leq (\| \nabla g(v_n) \| + \| \nabla g(u_{n+1}) \|)(\| u \| + \| u_{n+1} \|) + | D_g(u, v_n) | + | D_g(u, u_{n+1}) |.$$

(3.13)

By Remark 2.1(3), we find that $\nabla g$ is uniformly continuous on bounded subsets. By combining this with the boundedness of $\{u_n\},$ $\{D_g(u, u_n)\},$ $\{D_g(u, v_n)\},$ $\{v_n\}$ and (3.13), we find that the sequence $\{D_g(u_{n+1}, v_n)\}$ is bounded. It follows from (3.12), $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \xi_n = 0,$ we conclude that $\lim_{n\to\infty} D_g(u_{n+1}, v_{n+1}) = 0.$ Then, by Proposition 2.10 and the boundedness of $\{v_{n+1}\},$ we find that $\lim_{n\to\infty} \| u_{n+1} - v_{n+1} \| = 0.$ By combining this with (3.11), we have $\lim_{n\to\infty} \| v_{n+1} - u_n \| = 0.$ Since $g$ is uniformly continuous and $\nabla g$ is uniformly continuous on bounded subsets by Remark 2.1, we find that

$$\lim_{n\to\infty} \| g(v_{n+1}) - g(u_n) \| = \lim_{n\to\infty} \| \nabla g(v_{n+1}) - \nabla g(u_n) \| = 0. \tag{3.14}$$

From $v_{n+1} = H^M(\overline{w}_n),$ and using Lemma 2.23(3) and (3.4), we find that

$$D_g(v_{n+1}, \overline{w}_n) = D_g(H^M(\overline{w}_n), \overline{w}_n) \leq D_g(u, \overline{w}_n) - D_g(u, v_{n+1})$$
$$\leq a_n D_g(u, v_n) + (1 - a_n) D_g(u, u_n) + \xi_n - D_g(u, v_{n+1})$$
$$\leq a_n [D_g(u, v_n) - D_g(u, u_n)] + [D_g(u, u_n) - D_g(u, v_{n+1})] + \xi_n. \tag{3.15}$$

Furthermore, by the definition of the Bregman distance, we have

$$| D_g(u, u_n) - D_g(u, v_{n+1}) |$$
$$= | [g(u) - g(u_n) - \langle \nabla g(u_n), u - u_n \rangle] - [g(u) - g(v_{n+1}) - \langle \nabla g(v_{n+1}), u - v_{n+1} \rangle] |$$
$$\leq | g(v_{n+1}) - g(u_n) | + \| \nabla g(v_{n+1}) \| . \| u_n - v_{n+1} \| + \| \nabla g(v_{n+1}) - \nabla g(u_n) \| . \| u - u_n \|. \tag{3.16}$$

Then, by using (3.14), (3.16), the boundedness of $\{u_n\}, \{v_n\}, \{\nabla g(v_n)\}$ and the boundedness of $\mathcal{F},$ we have

$$\lim_{n\to\infty} | D_g(u, u_n) - D_g(u, v_{n+1}) | = 0.$$

(3.17)

It follows from (3.15), (3.17), the boundedness of $\{D_g(u, u_n)\},$ $\{D_g(u, v_n)\}$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \xi_n = 0$ that

$$\lim_{n\to\infty} D_g(v_{n+1}, \overline{w}_n) = 0. \tag{3.18}$$

Moreover, we conclude from (3.4) and the boundedness of $\{D_g(u, u_n)\},$ $\{D_g(u, v_n)\}$ that $\{D_g(u, \overline{w}_n)\}$ is bounded. By combining this with the fact that $\nabla g^*$ is bounded on bounded subsets and using Proposition 2.13, we find that $\{\overline{w}_n\}$ is bounded. Then, by using Proposition 2.10 and (3.18), we obtain that

$$\lim_{n\to\infty} \| v_{n+1} - \overline{w}_n \| = 0. \tag{3.19}$$

It follows from $\lim_{n\to\infty} \| v_{n+1} - u_n \| = 0$ and (3.19) that $\lim_{n\to\infty} \| u_n - \overline{w}_n \| = 0.$ By combining this with the uniform continuous of $g,$ the boundedness on bounded subsets of $\nabla g$ and the inequality

$$| D_g(u, \overline{w}_n) | \leq | g(u_n) - g(\overline{w}_n) | + \| \nabla g(\overline{w}_n) \| . \| u_n - \overline{w}_n \|,$$

we find that $\lim_{n\to\infty} D_g(u_n, \overline{w}_n) = 0.$ Therefore, by definition of $i_n$, we obtain $\lim_{n\to\infty} D_g(u_n, w_n^{(i)}) = 0.$ By using the arguments as in the proof of (3.19), we find that $\lim_{n\to\infty} \| u_n - w_n^{(i)} \| = 0.$ By combining this with the uniform continuous on bounded subsets of $\nabla g,$ we get

$$\lim_{n\to\infty} \| \nabla g(u_n) - \nabla g(w_n^{(i)}) \| = 0. \tag{3.20}$$

Note that $\nabla g = (\nabla g^*)^{-1}.$ Then we have

$$\| \nabla g(u_n) - \nabla g(w_n^{(i)}) \| = \| \nabla g(u_n) - a_n \nabla g(v_n) + (1 - a_n) \nabla g(S_i^n u_n) \|$$
$$= \| (1 - a_n)[\nabla g(u_n) - \nabla g(S_i^n u_n)] + a_n[\nabla g(u_n) - \nabla g(v_n)] \|$$
$$\geq (1 - a_n) \| \nabla g(u_n) - \nabla g(S_i^n u_n) \| - a_n \| \nabla g(u_n) - \nabla g(v_n) \|.$$

This leads to

$$(1 - a_n) \| \nabla g(u_n) - \nabla g(S_i^n u_n) \| \leq \| \nabla g(u_n) - \nabla g(w_n^{(i)}) \| + a_n \| \nabla g(u_n) - \nabla g(v_n) \|. \tag{3.21}$$

It follows from (3.20), (3.21), the boundedness of $\{\nabla g(u_n)\}, \{\nabla g(v_n)\}$ and $\lim_{n\to\infty} \alpha_n = 0$ that

$$\lim_{n\to\infty} \| \nabla g(u_n) - \nabla g(S_i^n u_n) \| = 0. \tag{3.22}$$

Then we conclude from (3.22) and the uniform continuous on bounded subsets of $\nabla g^*$ that

$$\lim_{n\to\infty} \| u_n - S_i^n u_n \| = 0. \tag{3.23}$$

By combining (3.23) and $\lim_{n\to\infty} u_n = p,$ we get that $\lim_{n\to\infty} S_i^n u_n = p$ for each $i \in I.$ Therefore, by combining this with the uniformly asymptotically regular property of $S_i$ and the following inequality

$$\| S_i^{n+1} u_n - p \| \leq \| S_i^{n+1} u_n - S_i^n u_n \| + \| S_i^n u_n - p \|,$$

we find that $\lim_{n\to\infty} S_i^{n+1} u_n = p$ and hence $\lim_{n\to\infty} S(S_i^n u_n) = p.$ By the closedness of $S_i,$ we have $S_i p = p.$ It means that $p \in \bigcap_{i\in I} F(S_i).$

Next, we prove that $p \in \bigcap_{k\in J} GMEP(f_k, \varphi_k, \psi_k).$ Indeed, for each $k \in K,$ by Lemma 2.24, we have

$$f_k(H^k(\overline{w}_n), v) + \varphi_k(v) + \langle \psi_k(H^k(\overline{w}_n)), v - H^k(\overline{w}_n) \rangle + \langle \nabla g(H^k(\overline{w}_n)) - \nabla g(H^{k-1}(\overline{w}_n)), v - H^k(\overline{w}_n) \rangle$$
$$\geq \varphi_k(H^k(\overline{w}_n)), \forall v \in \Omega.$$

By the condition (C2), This leads to

$$\varphi_k(v) - \varphi_k(H^k(\overline{w}_n)) + \langle \psi_k(H^k(\overline{w}_n)), v - H^k(\overline{w}_n) \rangle + \langle \nabla g(H^k(\overline{w}_n)) - \nabla g(H^{k-1}(\overline{w}_n)), v - H^k(\overline{w}_n) \rangle$$
$$\geq -f_k(H^k(\overline{w}_n), v) \geq f_k(v, H^k(\overline{w}_n)), \forall v \in \Omega.$$
(3.24)

Furthermore, by Lemma 2.24, (3.4) and the fact that $H^k$ is a Bregman quasi-nonexpansive mapping for each $k \in K,$ we find that

$$D_g(H^k(\overline{w}_n), \overline{w}_n) \leq D_g(u, H^{k-1}(\overline{w}_n)) - D_g(u, H^k(\overline{w}_n))$$
$$\leq D_g(u, H^{k-2}(\overline{w}_n)) - D_g(u, H^k(\overline{w}_n)) \leq \dots$$
$$\leq D_g(u, \overline{w}_n) - D_g(u, H^k(\overline{w}_n))$$
$$\leq a_n D_g(u, v_n) + (1 - a_n) D_g(u, u_n) + \xi_n - D_g(u, H^k(\overline{w}_n)). \tag{3.25}$$

Similarly, since $H^M$ is a Bregman quasi-nonexpansive mapping, we get that

$$D_g(u, v_{n+1}) = D_g(u, H^M(\overline{w}_n)) \leq D_g(u, H^k(\overline{w}_n)). \tag{3.26}$$

It follows from (3.25) and (3.26), we have

$$D_g(H^k(\overline{w}_n), \overline{w}_n) \leq a_n D_g(u, v_n) + (1 - a_n) D_g(u, u_n) + \xi_n - D_g(u, v_{n+1})$$
$$= a_n[D_g(u, v_n) - D_g(u, u_n)] + [D_g(u, u_n) - D_g(u, v_{n+1})] + \xi_n.$$

By combining this with (2.17), the boundedness of $\{D_g(u, u_n)\}$, $\{D_g(u, v_n)\}$ and $\lim_{n \to \infty} a_n = \lim_{n \to \infty} \xi_n = 0$ that

$$\lim_{n \to \infty} D_g(H^k(\overline{w}_n), \overline{w}_n) = 0. \tag{3.27}$$

By Proposition 2.10 and the boundedness of $\{\overline{w}_n\}$, we find that

$$\lim_{n \to \infty} \| H^k(\overline{w}_n) - \overline{w}_n \| = 0. \tag{3.28}$$

Note that

$$\| H^k(\overline{w}_n) - p \| \leq \| H^k(\overline{w}_n) - \overline{w}_n \| + \| \overline{w}_n - u_n \| + \| u_n - p \|. \tag{3.29}$$

It follows from (3.28), (3.29), $\lim_{n \to \infty} \| u_n - \overline{w}_n \| = 0$ and $\lim_{n \to \infty} u_n = p$ that $\lim_{n \to \infty} H^k(\overline{w}_n) = p$. Similarly, we find that $\lim_{n \to \infty} H^{k-1}(\overline{w}_n) = p$. Therefore, $\lim_{n \to \infty} \| H^k(\overline{w}_n) - H^{k-1}(\overline{w}_n) \| = 0$. By combining with the uniform continuous on bounded subsets of $\nabla g$, we have $\lim_{n \to \infty} \| \nabla g(H^k(\overline{w}_n)) - \nabla g(H^{k-1}(\overline{w}_n)) \| = 0$. This implies that $\lim_{n \to \infty} | \langle \nabla g(H^k(\overline{w}_n)) - \nabla g(H^{k-1}(\overline{w}_n)), v - H^k(\overline{w}_n) \rangle | = 0$. By combining this with $\lim_{n \to \infty} H^k(\overline{w}_n) = p$, the lower semi-continuous property of $\varphi_k$, the continuity of $\psi_k$, we find that

$$\varphi_k(v) - \varphi_k(p) + \langle \psi_k(p), v - p \rangle \geq f_k(v, p), \forall v \in \Omega$$

and hence for all $v \in \Omega$, we have

$$f_k(v, p) + \langle \psi_k(p), p - y \rangle + \varphi_k(p) - \varphi_k(v) \leq 0. \tag{3.30}$$

For all $t \in (0, 1]$, put $v_t = tv + (1 - t)p$. Since $v, p \in \Omega$ and $\Omega$ is convex, we have $v_t \in \Omega$. Thus, replacing $v$ by $v_t$ in (3.30), we obtain

$$f_k(v_t, p) + \langle \psi_k(p), p - v_t \rangle + \varphi_k(p) - \varphi_k(v_t) \leq 0. \tag{3.31}$$

Then, by using the condition (C1), the convexity in the second variable of $f$, the convexity of $\varphi$ and (3.31), we have

$$0 = f_k(v_t, v_t)$$
$$= f_k(v_t, v_t) + \langle \psi_k(p), v_t - v_t \rangle + \varphi_k(v_t) - \varphi_k(v_t)$$
$$\leq t f_k(v_t, v) + (1 - t) f_k(v_t, p) + t \langle \psi_k(p), v - v_t \rangle + (1 - t) \langle \psi_k(p), p - v_t \rangle + t \varphi_k(v) + (1 - t) \varphi_k(p) - \varphi_k(v_t)$$
$$= t[f_k(v_t, v) + \langle \psi_k(p), v - v_t \rangle + \varphi_k(v) - \varphi_k(v_t)] + (1 - t)[f_k(v_t, p) + \langle \psi_k(p), p - v_t \rangle + \varphi_k(p) - \varphi_k(v_t)]$$
$$\leq t[f_k(v_t, v) + \langle \psi_k(p), y - v_t \rangle + \varphi_k(v) - \varphi_k(v_t)].$$

Since by $t \in (0, 1]$, we have $f_k(v_t, v) + \langle \psi_k(p), v - v_t \rangle + \varphi_k(v) - \varphi_k(v_t) \geq 0$. Letting $t \to 0^+$ and using the condition (C3), we have $f_k(p, v) + \langle \psi_k(p), y - p \rangle + \varphi_k(v) - \varphi_k(p) \geq 0$. This leads to $p \in GMEP(f_k, \varphi_k, \psi_k)$ for each $k \in K$ and hence $p \in \bigcap_{k \in K} GMEP(f_k, \varphi_k, \psi_k)$. Therefore,

$$p \in \mathcal{F} = (\bigcap_{i \in I} F(S_i)) \cap (\bigcap_{k \in K} GMEP(f_k, \varphi_k, \psi_k)).$$

**Step 6.** *We claim that* $p = P_{\mathcal{F}}^g(u_1)$. Indeed, since $u_{n+1} = P_{\Omega_{n+1}}^g(u_1)$, by Proposition 2.10, we have

$$\langle \nabla g(u_1) - \nabla g(u_{n+1}), u_{n+1} - v \rangle \geq 0 \tag{3.32}$$

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for all $v \in \Omega_{n+1}$.       Let $u \in \mathcal{F}$. Since $\mathcal{F} \subset \Omega_{n+1}$, we have $u \in \Omega_{n+1}$. By choosing $v = u$ in (3.32), we get

$$\langle \nabla g(u_1) - \nabla g(u_{n+1}), u_{n+1} - u \rangle \geq 0.$$

Taking the limit in the above inequality as $n \to \infty$, using $\lim_{n\to\infty} u_n = p$ and the uniform continuous on bounded subsets of $\nabla g$, we find that $\langle \nabla g(u_1) - \nabla g(p), p - u \rangle \geq 0$ for all $u \in \mathcal{F}$. By Proposition 2.11, we find that $p = P_{\mathcal{F}}^g(u_1)$.

In Theorem 3.1, by choosing $S_i = S$ for all $i \in I$ and $k \in K = \{1\}$, $f_1 = f, \varphi_1 = \varphi, \psi_1 = \psi$, we obtain the following convergence result for a Bregman totally quasi-asymptotically nonexpansive mapping and a generalized mixed equilibrium problem in reflexive Banach spaces.

**Corollary 3.2.** *Assume that $W$ is a real reflexive Banach space, and $\Omega$ is a nonempty, closed and convex subset of $W$, and $g : W \to \mathbb{R}$ is Legendre, strongly coercive on $W$, and $g$ is bounded, totally convex, uniformly Fréchet differentiable on bounded subsets of $W$. Let $f, \varphi, \psi$ satisfy the conditions (C1)-(C6) and $S : \Omega \to \Omega$ be a closed, uniformly asymptotically regular and Bregman totally quasi-asymptotically nonexpansive mapping with nonnegative real sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfying $\lim_{n\to\infty} \alpha_n = 0$, $\lim_{n\to\infty} \beta_n = 0$ and strictly increasing continuous function $\zeta : [0,\infty) \to [0,\infty)$ with $\zeta(0) = 0$ such that $\mathcal{F} = F(S) \cap GMEP(f, \varphi, \psi)$ is nonempty and bounded. Let $\{u_n\}$ be a sequence generated by*

$$\begin{cases} u_1, v_1 \in \Omega, \Omega_1 = \Omega \\ w_n = \nabla g^*[a_n \nabla g(v_n) + (1 - a_n)\nabla g(S^n u_n)] \\ v_{n+1} = \mathrm{Res}_{f,\varphi,\psi}^g(w_n) \\ \Omega_{n+1} = \{u \in \Omega_n : D_g(u, v_{n+1}) \leq a_n D_g(u, v_n) + (1 - a_n)D_g(u, u_n) + \xi_n\} \\ u_{n+1} = P_{\Omega_{n+1}}^g(u_1), \forall n \geq 2 \end{cases}$$

*where $\xi_n = (1 - a_n)[\alpha_n \sup\{\zeta(D_g(u, u_n)) : u \in \mathcal{F}\} + \beta_n]$, the function $\mathrm{Res}_{f,\varphi,\psi}^g$ is defined as in Lemma 2.23 and $\{a_n\} \subset [0,1]$ such that $\lim_{n\to\infty} a_n = 0$. Then the sequence $\{u_n\}$ strongly converges to $p = P_{\mathcal{F}}^g(u_1)$.*

Note that if the Banach space $W$ is uniformly smooth, then $W$ is smooth and reflexive. By using Remark 2.7, from Theorem 3.1, we obtain the following convergence result for a finite family of totally quasi-$\phi$-asymptotically nonexpansive mappings and a system of generalized mixed equilibrium problems in uniformly smooth and strictly convex Banach spaces.

**Corollary 3.3.** *Assume that $W$ is a uniformly smooth and strictly convex, and $\Omega$ is a nonempty, closed and convex subset of $W$. For each $k \in K := \{1,2,...,M\}$ with $M \in \mathbb{N}$, the functions $f_k, \varphi_k, \psi_k$ satisfy the conditions (C1)-(C6). For each $i \in I$, $S_i : \Omega \to \Omega$ is a closed, uniformly asymptotically regular and totally quasi-$\phi$-asymptotically nonexpansive mapping with nonnegative real sequences $\{\alpha_n^{(i)}\}$ and $\{\beta_n^{(i)}\}$ satisfying $\lim_{n\to\infty} \alpha_n^{(i)} = \lim_{n\to\infty} \beta_n^{(i)} = 0$ and strictly increasing continuous function $\zeta^{(i)} : [0,\infty) \to [0,\infty)$ with $\zeta^{(i)}(0) = 0$ such that $\mathcal{F} = (\bigcap_{i \in I} F(S_i)) \cap (\bigcap_{k \in K} GMEP(f_k, \varphi_k, \psi_k))$ is nonempty and bounded. Let $\{u_n\}$ be a sequence generated by*

$$\begin{cases} u_1, v_1 \in \Omega, \Omega_1 = \Omega \\ w_n^{(i)} = J^{-1}[a_n J(v_n) + (1 - a_n)J(S_i^n u_n)] \\ i_n = \arg\max\{\phi(u_n, w_n^{(i)}) : i \in I\}, \overline{w}_n = w_n^{(i_n)} \\ v_{n+1} = H^M(\overline{w}_n) \\ \Omega_{n+1} = \{u \in \Omega_n : \phi(u, v_{n+1}) \le a_n \phi(u, v_n) + (1 - a_n)\phi(u, u_n) + \xi_n\} \\ u_{n+1} = \prod_{\Omega_{n+1}}(u_1), \forall n \ge 2 \end{cases}$$

where $\xi_n = (1 - a_n)[\alpha_n \sup\{\zeta(\phi(u, u_n)) : u \in \mathcal{F}\} + \beta_n]$, the function $R_k := \operatorname{Res}^g_{f_k, \varphi_k, \psi_k}$ is defined as in Lemma 2.24, where $\nabla g$ is replaced by $J$, $H^k = R_k(R_{k-1}(R_{k-2}(...(R_1))))$ for all $k \in K$, $\{a_n\} \subset [0,1]$ such that $\lim_{n \to \infty} a_n = 0$. Then the sequence $\{u_n\}$ strongly converges to $p = \prod_{\mathcal{F}}(u_1)$.

From Corollary 3.3, by choosing $S_i = S$ for all $i \in I$ and $k \in K = \{1\}$, $f_1 = f, \varphi_1 = \varphi, \psi_1 = \psi$, we obtain the following convergence result for a totally quasi-$\phi$-asymptotically nonexpansive mapping and a generalized mixed equilibrium problems in uniformly smooth and strictly convex Banach spaces.

**Corollary 3.4.** *Assume that $W$ is a uniformly smooth and strictly convex, and $\Omega$ is a nonempty, closed and convex subset of $W$. Let the functions $f, \varphi, \psi$ satisfy the conditions (C1)-(C6) and $S : \Omega \to \Omega$ be a closed, uniformly asymptotically regular and totally quasi-$\phi$-asymptotically nonexpansive mapping with nonnegative real sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfying $\lim_{n \to \infty} \alpha_n = 0$, $\lim_{n \to \infty} \beta_n = 0$ and strictly increasing continuous function $\zeta : [0, \infty) \to [0, \infty)$ with $\zeta(0) = 0$ such that $\mathcal{F} = F(S) \cap GMEP(f, \varphi, \psi) \neq \varnothing$ and bounded. Let $\{u_n\}$ be a sequence generated by*

$$\begin{cases} u_1, v_1 \in \Omega, \Omega_1 = \Omega \\ w_n = J^{-1}[a_n J(v_n) + (1 - a_n)J(S^n u_n)] \\ v_{n+1} = \operatorname{Res}^g_{f, \varphi, \psi}(w_n) \\ \Omega_{n+1} = \{u \in \Omega_n : \phi(u, v_{n+1}) \le a_n \phi(u, v_n) + (1 - a_n)\phi(u, u_n) + \xi_n\} \\ u_{n+1} = \prod_{\Omega_{n+1}}(u_1), \forall n \ge 2 \end{cases}$$

*where $\xi_n = (1 - a_n)[\alpha_n \sup\{\zeta(\phi(u, u_n)) : u \in \mathcal{F}\} + \beta_n]$, the function $\operatorname{Res}^g_{f, \varphi, \psi}$ is defined as in Lemma 2.24, where $\nabla g$ is replaced by $J$, $\{a_n\} \subset [0,1]$ such that $\lim_{n \to \infty} a_n = 0$. Then the sequence $\{u_n\}$ strongly converges to $p = \prod_{\mathcal{F}}(u_1)$.*

**Remark 3.5.** (1) *It follows from Remark 2.21 that each Bregman quasi-asymptotically nonexpansive mapping is a Bregman totally quasi-asymptotically nonexpansive mapping. Therefore, the conclusions in Theorem 3.1, Corollary 3.2 are hold when a Bregman totally quasi-asymptotically nonexpansive mapping is replaced by a Bregman quasi-asymptotically nonexpansive mapping. Similary, the conclusions in Corollary 3.3 and Corollary 3.4 are hold when a totally quasi-$\phi$-asymptotically nonexpansive mapping is replaced by a quasi-$\phi$-asymptotically nonexpansive mapping.*

(2) *Theorem 3.1 is an improvement of ([8], Theorem 4.1) from a generalized equilibrium and a Bregman relatively nonexpansive mapping to a finite family of Bregman totally quasi-asymptotically nonexpansive mappings and a system of generalized mixed equilibrium problems in reflexive Banach spaces.*

(3) *Corollary 3.3 and Corollary 3.4 are improvements of ([7], Theorem 3.1) from an equilibrium problem and a fixed point problem for an asymptotically quasi-$\phi$-nonexpansive mapping in intermediate sense to generalized mixed equilibrium problems and totally quasi-$\phi$-asymptotically nonexpansive mappings in uniformly smooth and strictly convex Banach spaces.*

Finally, we give an example to illustrate for Theorem 3.1. In the following example, the Bregman distance $D_g$ is not a norm and the Bregman projection is not a the generalized projection in smooth Banach spaces. Therefore, ([7], Theorem 3.1) cannot be applicable to the given mappings in Example 3.6.

**Example 3.6.** Let $W = \mathbb{R}$, $\Omega = [0,1]$, $g(u) = u^4$ for all $u \in \mathbb{R}$. Then for all $u, w \in \mathbb{R}$, we have

$$\nabla g(u) = 4u^3, \qquad\qquad g^*(w) = 3\left(\frac{w}{4}\right)^{4/3}, \qquad\qquad \nabla g^*(w) = \left(\frac{w}{4}\right)^{1/3},$$

$$D_g(u,v) = u^4 - v^4 - 4v^3(u-v) = u^4 + 3v^4 - 4uv^3.$$

For each $i \in I = \{1,2\}$, we consider $S_i(u) = \dfrac{u}{2^i}$ for all $u \in \Omega$. We have $F(S_i) = \{0\}$. Therefore, for $w \in F(S_i)$ and $u \in \Omega$, we obtain

$$D_g(w, S_i^n u) = D_g(0, S_i^n u) = 3(S_i^n u)^4 = 3\left(\frac{u}{2^{ni}}\right)^4 \le 3u^4 = D_g(0,u) = D_g(w,u).$$

This proves that $S_i$ is a Bregman totally quasi-asymptotically nonexpansive mapping with $\alpha_n^{(i)} = \beta_n^{(i)} = 0$ for all $n \in \mathbb{N}^*$. For each $k \in K = \{1,2\}$, we consider

$$f_k(u,v) = -u^2 + uv, \ \varphi_k(u) = u^2, \ \psi_k(u) = u \text{ for all } u, v \in \Omega. \text{ Then, by directly checking, we find that}$$

$f_k, \varphi_k, \psi_k$ satisfies the conditions (C1)-(C6). Now, we find the formula of $w = \text{Res}_{f_k,\varphi_k,\psi_k}^g(u)$ for $u \in W, w \in \Omega$ as in Lemma 2.24. Indeed, $w = \text{Res}_{f_k,\varphi_k,\psi_k}^g(u)$ if

$$\text{Res}_{f_k,\varphi_k,\psi_k}^g(u) = \{w \in \Omega : f_k(w,v) + \varphi_k(v) + \langle \psi_k(w), v - w \rangle + \langle \nabla g(w) - \nabla g(u), v - w \rangle \ge \varphi_k(w), \forall v \in \Omega\}. \tag{3.33}$$

By substituting $f_k, \varphi_k, \psi_k, \nabla g$ into (3.33) and by directly calculating, we get

$$v^2 + (2w + 4w^3 - 4u^3)v + 4u^3 w - 4w^4 - 3w^2 \ge 0.$$

Put $h(v) = v^2 + (2w + 4w^3 - 4u^3)v + 4u^3 w - 4w^4 - 3w^2$ for all $v \in \Omega$. Then $h(v)$ is a quadratic function and $\Delta = (4w + 4w^3 - 4u^3)^2$. We consider the following cases.

*Case 1.* $\Delta > 0$. Then the equation $h(v) = 0$ has two solutions: $v_1 = w$ and $v_2 = 4u^3 - 4w^3 - 3w$. In order to $h(v) \ge 0$ for all $v \in \Omega$, we have following two cases:

*Case 1.1.* $v_1 = 1$ and $v_1 < v_2$. Then $w = v_1 = 1$, $v_2 = 4u^3 - 7 > 1$, and hence $u > \sqrt[3]{2}$.

*Case 1.2.* $v_1 = 0$ and $v_2 < v_1$. Then $w = v_1 = 0$ and $v_2 = 4u^3 < 0$. This leads to $u < 0$.

*Case 2.* $\Delta \le 0$. Then $w + w^3 = u^3$ and $h(v) \ge 0$ for all $v \in \Omega$. From $w + w^3 = u^3$, we get

$$w = \frac{\left(\sqrt[3]{\sqrt{81u^6 + 12} + 9u^3}\right)^2 - \sqrt[3]{12}}{\sqrt[3]{18}\sqrt[3]{\sqrt{81u^6 + 12} + 9u^3}} := R(u).$$

Since $w \in \Omega$, we have $0 \le u^3 \le 2$ and hence $0 \le u \le \sqrt[3]{2}$. Therefore,

$$\text{Res}_{f_k,\varphi_k,\psi_k}^g(u) = w = \begin{cases} 0 & \text{if } u < 0 \\ R(u) & \text{if } 0 \le u \le \sqrt[3]{2} \\ 1 & \text{if } u > \sqrt[3]{2}. \end{cases}$$

By taking $a_n = \dfrac{1}{n+1}$, we have $\lim\limits_{n \to \infty} a_n = 0$. By the above, all assumptions in Theorem 3.1 are satisfied with the given functions $f_k, \varphi_k, \psi_k, S_i$. Therefore, by Theorem 3.1, the sequence $\{u_n\}$ which is defined by (3.1) converges to $0 \in \mathcal{F} = (\bigcap\limits_{i \in I} F(S_i)) \cap (\bigcap\limits_{k \in K} GMEP(f_k, \varphi_k, \psi_k))$.

Let $u_1 = 0.5, v_1 = 0.6, \Omega_1 = \Omega$. It follows from directly calculating that the iterative process (3.1) becomes the following iteration.

$$
\begin{cases}
w_n^{(i)} = \dfrac{[4a_n v_n^3 + 4(1-a_n)\dfrac{u_n^3}{2^{3ni}}]^{1/3}}{\sqrt[3]{4}}, \forall i = 1,2 \\
i_n = \arg\max\{u_n^4 + 3(w_n^{(i)})^4 - 4u_n(w_n^{(i)})^3 : i = 1,2\} \\
\bar{w}_n = w_n^{(i_n)}, v_{n+1} = R(R(\bar{w}_n)) \\
u_{n+1} = \dfrac{3a_n v_n^4 + 3(1-a_n)u_n^4 - 3v_{n+1}^4}{4a_n v_n^3 + 4(1-a_n)u_n^3 - 4v_{n+1}^3}, \forall n \geq 2.
\end{cases}
\tag{3.34}
$$

By using Scilab-6.0 software for calculating the numerical result of the iteration (3.34), we find that $\{u_n\}$ approximates to 0 when $n = 54$ and $\| u_n - 0 \| < 10^{-7}$. The convergence behavior of the iteration (3.34) is presented by the following figure.
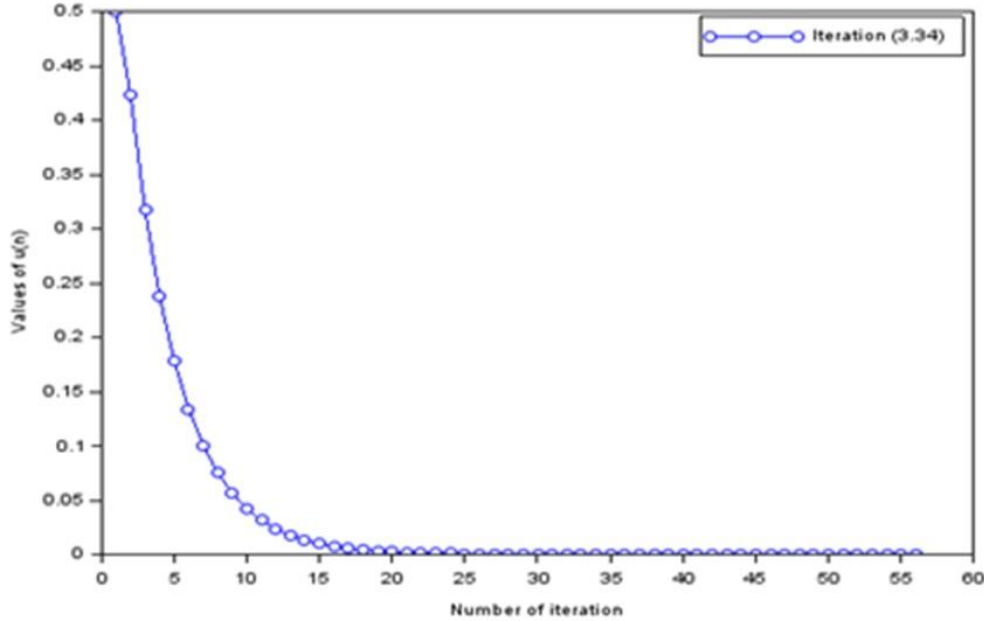


**Figure 1:** The convergence behavior of the iteration $\{u_n\}$ generated by (3.34) to 0.

# 4 Conclusion

In this paper, a hybrid iterative method is proposed for solving a finite system of generalized mixed equilibrium problems and common fixed point problem of a finite family of Bregman totally quasi-asymptotically nonexpansive mappings reflexive Banach spaces. Furthermore, a strong

convergence theorem for the proposed iteration is proved in reflexive Banach spaces. As application, we obtain some convergence results for generalized mixed equilibrium problems and totally quasi-$\phi$-asymptotically nonexpansive mappings in uniformly smooth and strictly convex Banach spaces. In addition, a numerical example is given to illustrate for the obtained results. These results generalize and extend the main results in [7, 8].

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# Conflict of Interest

There are no potential conflicts of interest with respect to the research, authorship, and publication of this article.

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