Derivational modal logic of real line with difference modality

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Abstract

We study derivational modal logic of real line with difference modality and prove that it has finite model property but does not have finite axiomatization.

Definition 1. Formulas are constructed from set of variables Prop, constant $\bot$ and connective $\to$ and modal operators $\Box$ and $[\neq]$. The other connectives ($\neg$, $\vee$, $\wedge$) considered as short-cuts. Dual modalities $\Diamond$ and $\langle\neq\rangle$ are expressible in the following way $\Diamond A \rightleftharpoons \neg\Box\neg A$, $\langle\neq\rangle A \rightleftharpoons \neg[\neq]\neg A$. We will also use the following short-cuts $[\forall]\, A \rightleftharpoons [\neq]A \wedge A$, $\boxplus A = \Box A \wedge A$.

Definition 2. *Topological model* is a pair $M = (\mathfrak{X}, \theta)$, where $\mathfrak{X}$ is a topological space and $\theta : \text{Prop} \to 2^{\mathfrak{X}}$ is a *valuation on* $\mathfrak{X}$. *Truth of a formula A at a point $x \in \mathfrak{X}$* ($M, x \models A$) is defined by induction as usual:

$$
\begin{array}{lllll}
1) & M, x \models p & \iff & x \in \theta(p) \\
2) & M, x \models A \to B & \iff & (M, x \models A \Rightarrow M, x \models B) \\
3) & M, x \models \Box A & \iff & \exists U(x) \forall y \in U(x)\, (y \neq x \Rightarrow M, y \models A) \\
4) & M, x \models [\neq]A & \iff & \forall y\, (y \neq x \Rightarrow M, y \models A)
\end{array}
$$

where $p \in \text{Prop}$ and $U(x)$ is a neighbourhood of x. Formula is called dd-valid in a space $\mathfrak{X}$ (or class of spaces $\mathcal{C}$) if it is true at all points in all models on $\mathfrak{X}$ (in all spaces from $\mathcal{C}$).

Definition 3. *Kripke frame F* is a triple (W, R, R_D), where $W \neq \varnothing$ and $R, R_D \subseteq W \times W$.

Definition 4. *Kripke model on a (Kripke) frame F* is a pair $M = (F, \theta)$, where $\theta : \text{Prop} \to 2^W$ is a valuation on F. The truth of a formula at a point in a Kripke model defines as usual in particular

$$
\begin{array}{lll}
M, x \models \Box\phi & \iff & \forall y (xRy \Rightarrow M, y \models \phi), \\
M, x \models [\neq]\phi & \iff & \forall y (xR_D y \Rightarrow M, y \models \phi).
\end{array}
$$

Definition 5. A *(normal) 2-modal logic* is a set of modal formulas containing the classical tautologies, axiom $\Box(p \to q) \to (\Box p \to \Box q)$ and the same for $[\neq]$, and closed under the standard inference rules: Modus Ponens ($A,\ A \to B/B$), Necessitation ($A/\Box_i A$), and Substitution ($A(p_j)/A(B)$).

K_2 stands for the *minimal 2-modal logic*. An 2-modal logic containing a certain 2-modal logic $\mathbf{L}$ is called an *extension* of $\mathbf{L}$, or a $\mathbf{L}$*-logic*. The minimal $\mathbf{L}$-logic containing a set of 2-modal formulas Γ is denoted by $\mathbf{L} + \Gamma$.

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*The work on this paper is patialy sposored by RFBR grants 11-01-00958-a and 11-01-93107-a

Consider the following axioms

$$
\begin{array}{ll}
(B_D) & p \to [\neq]\langle\neq\rangle p \\
(4^0_D) & (p \wedge [\neq]p) \to [\neq][\neq]p \\
(4^0_\Box) & \boxplus p \to \Box\Box p \\
(4_\Box) & \Box p \to \Box\Box p \\
(D_\Box) & [\neq]p \to \Box p \\
(AT_1) & [\neq]p \to [\neq]\Box p \\
(DS) & \Diamond\top \\
(AC) & [\forall]\,(\Box p \vee \Box\neg p) \to [\forall]\,p \vee [\forall]\,\neg p \\
(Ku_2) & \Box \bigvee_{k=0}^{2} \boxplus Q_k \to \bigvee_{k=0}^{2} \Box\neg Q_k,
\end{array}
$$

where $Q_1 = q_1 \wedge q_2$, $Q_2 = q_1 \wedge \neg q_2$ and $Q_3 = \neg q_1$

We introduce the following logics

$$
\begin{array}{rcl}
\mathsf{D4} & = & \mathsf{K}\,\{4_\Box, D_\Box\}\,, \\
\mathsf{K4^0D} & = & \mathsf{K}_2 + \left\{B_D, 4^0_D, D_\Box, 4^0_\Box\right\}, \\
\mathsf{LC}_2 & = & \mathsf{K4^0D} + \{4_\Box, AT_1, DS, AC, Ku_2\}\,.
\end{array}
$$

Logic of a class $\mathcal{C}$ of Kripke frames is the set of all formulas valid in all frames from $\mathcal{C}$ (notation: $L(\mathcal{C})$). A frame F called an *L-frame* if $\forall A \in L(F \models A)$. dd-logic of a space $\mathfrak{X}$ (or a class of spaces $\mathcal{C}$) is the set of all dd-valid formulas in this space $\mathfrak{X}$ (or in $\mathcal{C}$) (notation: $Ld_{\neq}(\mathcal{C})$).

Theorem 1 ([3])**.** *Logic* $\mathsf{K4^0D}$ *is the dd-logic of all topological spaces.*

Definition 6. A topological space $\mathfrak{X}$ is called *locally connected* if every neighbourhood of any point x contains a connected neighbourhood of x. Locally connected $\mathfrak{X}$ is called *locally n-component* if for any connected neighbourhood U of any point x $U - \{x\}$ contains at most n connected components.

The flowing correspondences is well-known ([1], [5], [3]).

Axioms	Property of topological space
(DS)	density-in-itself
$(4_\Box)$	T_d
(AT_1)	T_1
(Ku_2)	locally 2-componentness
(AC)	connectedness

Real line $\mathbb{R}$ satisfy these properties so $Ld_{\neq}(\mathbb{R}) \supseteq \mathsf{LC}_2$. But in fact $Ld_{\neq}(\mathbb{R}) \neq \mathsf{LC}_2$ (see Theorem 4).

Let $F = (W, R, R_D)$ be an $\mathsf{K4^0D}$-frame. Then R is weakly transitive relation and $R_D \cup Id_W$ is universal. Put $\widehat{R} \leftrightharpoons R\circ R^{-1}$; $\widetilde{R}$ is the reflexive transitive closer of $\widehat{R}$. Then $\widetilde{R}$ is an equivalence. Frame F is called *connected*, if $\forall x \forall y (x\widetilde{R}y)$. $R^* \leftrightharpoons \widetilde{R\big|_{W^-}}$, where $W^- \leftrightharpoons \{w \in W \mid wR_Dw\}$. F *locally* 2*-component*, if for each x $R(x)$ intersects with at most 2 R^*-classes.

The following correspondences can be found in [3]

Axioms	Property of Kripke frame
(DS)	$\forall x(R(x) \cap R_D(x) \neq \varnothing)$
$(4_\Box)$	transitivity of R
(AT_1)	$\forall x(\neg xR_Dx \Rightarrow \neg\exists y(yRx \;\&\; y \neq x))$
(Ku_2)	locally 2-componentness
(AC)	connectedness

Definition 7. Let $F = (W, R, R_D)$ be a $\mathsf{K4^0D}$-frame and $\mathfrak{X}$ a topological space. An onto function $f : \mathfrak{X} \to F$ is called *dd-morphism* (notation: $f : \mathfrak{X} \to^{dd} F$) if for any $w \in W$: (1) $\mathbf{d}f^{-1}(w) = f^{-1}(R^{-1}(w))$, (2) if $\neg wR_D w$ then $f^{-1}(w)$ is a one-element-set (where $\mathbf{d}$ is the derivative operator on $\mathfrak{X}$).

Lemma 2. *[3] If $f : \mathfrak{X} \to^{dd} F$ and F is a finite $\mathbf{K4^0D}$-frame then $Ld_{\neq}(\mathfrak{X}) \subseteq L(F)$.*

In this paper by *graph* Γ we understand 1d simplicial complex or pseudograph; to be precise $\Gamma = (V, E, \varepsilon)$, where V is a set of *vertices*, E is a set of *edges*, $\varepsilon : E \to \{\{v_1, v_2\} \mid v_1, v_2 \in V\}$. $\varepsilon(e)$ are *endpoints* of edge e. *A walk* is a sequence $P = (v_0 e_1 v_1 \dots e_n v_n)$, where $\varepsilon(e_i) = \{v_{i-1}, v_i\}$ for each $i = 1, \dots, n$. A walk is *Eulerian* if it uses all edges of the graph precisely once. A graph is *traversable* if there is an Eulerian walk in it.

For a LC_2-frame $F = (W, R, R_D)$ we construct graph $\Gamma(F) = (V, W^\circ, \varepsilon)$ in the following way $W^\circ = \{w \mid \neg wR_D w\}$, $W^- = W - W^\circ$, $V = W^-/R^*$, $\varepsilon(x) = \{[w] \in V \mid [w] \cap R(x) \neq \varnothing\}$ (here $[x]$ is the R^*-equivalence class of x). Since $F \models DS \wedge AE_2$ then $\varepsilon(x)$ contains one or two points.

Lemma 3. *If $F = (W, R, R_D)$ is a finite LC_2-frame and $f : \mathbb{R} \to^{dd} F$, iff graph $\Gamma(F)$ is traversable.*

Proof. From left to right it was proved in [2].

From right to left we need to construct a dd-morphism. We use the result from from [4]. The author constructed a d-morphism from $\mathbb{R}$ onto any connected 2-component $\mathsf{D4}$-frame. Note that in this construction preimage of any irreflexive root point is a one-element-set.

Let $A = \{x \in W \mid \neg xRx\}$ and $B = \{x \in W \mid \neg xR_D x\}$. Due to axiom $D_\Box$ $B \subseteq A$. Since F is finite assume that $C = A - B = \{c_1, \dots, c_k\}$. If $C = \varnothing$ then $F' = F$.

Otherwise, we define $W' = W \cup C'$, where $C' = \{c'_1, \dots, c'_k\}$ be a disjoint copy of C. R' is an extension of R, such that $R'(c'_i) = R(c_i)$ and R'_D is such that $R'_D \cup Id = W' \times W'$ and the only R'_D-irreflexive points are $A \cup C'$. Put $F' = (W', R', R'_D)$.

Let $g : W' \to W$ is identical on W and for any $i \in \{1, \dots, k\}$ $g(c'_i) = c_i$. It is easy to check that g is a p-morphism. Now we use [4] and construct d-morphism $h : \mathbb{R} \twoheadrightarrow^d F'$ and we put $f = g \circ h$. We left to the reader to check that $f : \mathbb{R} \to^{dd} F$. □

This Lemma allow us to prove that dd-logic of $\mathbb{R}$ is not finitely axiomatizable and even stronger theorem. A logic is called *n-axiomatizable* is it has an axiomatization which uses only n variables.

Theorem 4. *Logic $Ld_{\neq}(\mathbb{R})$ is not n-axiomatizable for any n.*

Basically the construction repeats the one from [2].

Let $\mathcal{C}_E$ be the set of all finite LC_2-frames, such that $\Gamma(F)$ is traversable and $L_E = L(\mathcal{C}_E)$.

By Lemmas 3 and 2 $Ld_{\neq}(\mathbb{R}) \subseteq L_E$. We can even prove

Theorem 5. *Logics $Ld_{\neq}(\mathbb{R})$ and L_E coincides.*

This theorem can be proved using topofiltration — an analogue of epifiltration. We are filtering topological model and get a finite LC_2-frame F such that $\Gamma(F)$ is traversable. Since the size of the frame has an upper bound

Corollary 6. *Logic $Ld_{\neq}(\mathbb{R})$ is decidable.*

References

[1] L. Esakia. Weak transitivity — a restitution (in russian). *Logical investigations*, 8:244–245, 2001.

[2] A. Kudinov. Topological modal logic of r with inequality. *Russian Mathematical Surveys*, 63:163–165, 2008.

[3] A. Kudinov and V. Shehtman. Derivationalmodallogicswiththedifferencemodality. To appearinspecialissueofTrendsinLogicVolume,Springer.

[4] J. Lucero-Bryan. The d-logic of the real line. *Journal of Logic and Computation*, 23(1):121–156, 2013.

[5] V. B. Shehtman. "Everywhere" and "here". *Journal of Applied Non-Classical Logics*, 9(2-3):369–380, 1999.