Lebesgue Constants and Optimal Node Systems via Symbolic Computations

Short Paper

Robert Vajda

University of Szeged, Szeged, Hungary
vajdar@math.u-szeged.hu

Abstract

Polynomial interpolation is a classical method to approximate continuous functions by polynomials. To measure the correctness of the approximation, Lebesgue constants are introduced. For a given node system $X^{(n+1)} = \{x_1 < \ldots < x_{n+1}\}\,(x_j \in [a,b])$, the Lebesgue function $\lambda_n(x)$ is the sum of the modulus of the Lagrange basis polynomials built on $X^{(n+1)}$. The Lebesgue constant Λ_n assigned to the function $\lambda_n(x)$ is its maximum over $[a,b]$. The Lebesgue constant bounds the interpolation error, i.e., the interpolation polynomial is at most $(1+\Lambda_n)$ times worse then the best approximation. The minimum of the Λ_n's for fixed n and interval $[a,b]$ is called the optimal Lebesgue constant Λ_n^*. For specific interpolation node systems such as the equidistant system, numerical results for the Lebesgue constants Λ_n and their asymptotic behavior are known [3, 7]. However, to give explicit symbolic expression for the minimal Lebesgue constant Λ_n^* is computationally difficult. In this work, motivated by Rack [5, 6], we are interested for expressing the minimal Lebesgue constants symbolically on $[-1,1]$ and we are also looking for the characterization of the those node systems which realize the minimal Lebesgue constants. We exploited the equioscillation property of the Lebesgue function [4] and used quantifier elimination and Groebner Basis as tools [1, 2]. Most of the computation is done in Mathematica [8].

Acknowledgement. The research of the author was partially supported by the HSRF (OTKA), grant number K83219.

References

[1] D. S. Arnon - G. E. Collins - S. McCallum, Cylindrical Algebraic Decomposition I: The Basic Algorithm. In: Caviness-Johnson (eds): Quantifier Elimination and Cylindrical Algebraic Decomposition, 136-151, Springer, 1998.

[2] B. Buchberger, Ein Algorithmus zum Auffinden der Basiselemente des Restklassenringes nach einem nulldimensionalen Polynomideal (An algorithm for finding the basis elements in the residue class ring modulo a zero dimensional polynomial ideal). PhD Thesis, Innsbruck, 1965.

[3] J. S. Hesthaven, From electrostatics to almost optimal nodal sets for polynomial interpolation in a simplex, SIAM J. Numer. Anal., Vol. 35, No. 2, 655-676, 1998.

[4] T. A. Kilgore, A characterization of the Lagrange interpolating projection with minimal Tchebycheff norm, J. Approx. Theory 24 (1978), no. 4, 273-288.

[5] H.-J. Rack, An example of optimal nodes for interpolation, International Journal of Mathematical Education in Science and Technology 15 (3): 355-357, 1984.

[6] H.-J. Rack, An example of optimal nodes for interpolation revisited, Advances in Applied Mathematics and Approximation Theory, Springer Proceedings in Mathematics & Statistics, Volume 41, 117-120, 2013.

[7] S. J. Simon, Lebesgue constants in polynomial interpolation, Annales Mathematicae et Informaticae 33: 109-123, 2006.

[8] Wolfram Research Inc., Mathematica, Version 9.0, Champaign, Illinois, 2012.