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Towards a Practical Decision Procedure for Uniform Interpolants of $\mathcal{EL}$-TBoxes – a Proof-Theoretic Approach*

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Abstract

We show how the problem of deciding the existence of uniform interpolants of TBoxes formulated in the Description Logic $\mathcal{EL}$ can be divided into three subproblems based on a characterisation of the logical difference between $\mathcal{EL}$-TBoxes. We propose a proof-theoretic decision procedure for subsumer interpolants of general TBoxes formulated in the Description Logic $\mathcal{EL}$, solving one of these subproblems. A subsumer interpolant of a TBox depends on a signature and on a concept name. It is a TBox formulated using symbols from the signature only such that, both, it follows from the original TBox and it exactly entails the subsumers formulated in the signature that follow from the concept name w.r.t. the original TBox. Our decision procedure first constructs a graph that exactly represents the part of original TBox that is describable using signature symbols only. Subsequently, it is checked whether a graph-representation of the original TBox can be simulated by the constructed graph, in which case a subsumer interpolant exists. We also evaluate our procedure by applying a prototype implementation on several biomedical ontologies.

1 Introduction

Ontologies are widely used to represent domain knowledge. They contain specifications of objects, concepts and relationships that are often formalised using a logic-based language over a vocabulary that is particular to an application domain. Description Logics [2] have been widely adopted as a basis for ontology languages, e.g., the Web Ontology Language (OWL) and its profiles are based on description logics. Numerous ontologies have been developed, in particular, in knowledge intensive areas such as the biological and the medical domain.

Ontologies are constantly developed, extended, corrected, and refined. As a result ontologies tend to become complex in structure and large in size. For instance, the GALEN ontology contains more than 20 000 term definitions, and the National Cancer Institute ontology consists of more than 60 000 term definitions. As the size of ontologies increases, being able to focus on relevant parts of an ontology by forgetting the knowledge about irrelevant or confidential terms has become an important concern. Forgetting in ontologies can be formalised by employing the notion of *predicate forgetting* and its dual *uniform interpolation*, both of which have been studied to a large extent in the area of logic and AI [3, 5, 8, 12, 13, 19, 24, 26, 27].

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The result of forgetting terms from an ontology yields a uniform interpolant, i.e. an ontology that is functionally equivalent regarding the terms that remain in the ontology. Several application scenarios for forgetting have been discussed including predicate hiding (concealing terms that are not intended for public use), exhibiting hidden relations between terms (making inferred relations between terms more explicit), and ontology versioning (ensuring that meaning of certain terms remains unaffected by update) [12,16,18].

Forgetting and uniform interpolation have also been studied extensively in the area of description logics. Algorithms for forgetting in expressive description logics have been proposed in [24,25,27], and algorithms for computing uniform interpolants based on resolution have been developed in [14,15,28]. However, uniform interpolants in such logics may not always exist, and none of these algorithms have been shown to be complete. This may not be surprising as such completeness results appear difficult to obtain. Deciding the existence of a uniform interpolant in $\mathcal{ALC}$ has been shown to be 2-ExpTime-complete. On the other hand, it has been shown for some description logics of the DL-Lite family that uniform interpolants always exist [13,26].

We consider ontologies formulated in the lightweight Description Logic $\mathcal{EL}$ [1]. The relevance of $\mathcal{EL}$ for ontologies is emphasised by the fact that many ontologies are largely, or even entirely, formulated in $\mathcal{EL}$. Uniform interpolation for $\mathcal{EL}$-TBoxes of a restricted form (a.k.a. terminologies) has been shown to be tractable [12]. The size of uniform interpolants for general $\mathcal{EL}$-TBoxes has been established to be 3-ExpTime in the worst case [21,22]. The problem for deciding the existence of uniform interpolants for $\mathcal{EL}$-TBoxes has been shown to be ExpTime-complete in [18] using a mixture of model-theoretic and automata-theoretic methods from [19].

In [18] the notion of an $\mathcal{EL}$-automaton is introduced to represent a Σ-uniform interpolant of an $\mathcal{EL}$-TBox. In a second step, Alternating Parity Tree Automata (APTA) are used to check whether the uniform interpolant can be expressed as an $\mathcal{EL}$-TBox using Σ-symbols only. However, automata-theoretic techniques do not necessarily facilitate the derivation of practical algorithms, and to the best of our knowledge, no approach that has been demonstrated to work in practice has been devised so far. To obtain a method that is easily implementable and that works in practice, we follow our proof-theoretic approach to the logical difference problem [6,7]. Applications of proof-theory methods have resulted in efficient implementations in many areas, e.g., the $\mathcal{EL}$-reasoner ELK [10] or the first-order theorem prover VAMPIRE [23].

In this paper, we first show how the problem of deciding the existence of uniform interpolants of $\mathcal{EL}$-TBoxes can be divided into three subproblems based on a characterisation of the logical difference between $\mathcal{EL}$-TBoxes [7]. Subsequently, we present a decision procedure for subsumer interpolants of $\mathcal{EL}$-TBoxes, solving one of these subproblems. A subsumer interpolant of a TBox $\mathcal{T}$ depends on a signature Σ and on a concept name A. It is a TBox $\mathcal{T}_\Sigma$ formulated using symbols from Σ only such that, both, it follows from $\mathcal{T}$ (i.e., $\mathcal{T} \models \mathcal{T}_\Sigma$) and it exactly entails the subsumers formulated in Σ that follow from A w.r.t. $\mathcal{T}$. The idea behind the decision procedure for the existence of subsumer interpolants is the following. We construct a TBox $\mathcal{T}^{\Sigma,A}_{\text{desc}}$ that describes the part of $\mathcal{T}$ relevant for entailing the Σ-subsumers of A w.r.t. $\mathcal{T}$ as complete as possible using Σ-consequences of $\mathcal{T}$ only. Since subsumer interpolants do not always exist, we then check whether $\mathcal{T}^{\Sigma,A}_{\text{desc}}$ captures all the Σ-subsumers of A w.r.t. $\mathcal{T}$.

We proceed as follows. In the next section, we define some preliminary notions. In Section 3, we formally introduce the notions of a uniform interpolant and of a subsumer interpolant of an $\mathcal{EL}$-TBox for a signature. We introduce our graph-based approach to characterising the necessary and sufficient conditions for the existence of subsumer interpolants. Moreover, we demonstrate the viability of our approach by showing evaluation results of applying a prototype implementation on several biomedical ontologies in Section 4.

2 Preliminaries

Let $\mathsf{N_C}$ and $\mathsf{N_R}$ be disjoint sets of concept names and role names, which we assume to be countably infinite. *$\mathcal{EL}$-concepts* C are built according to the grammar rule $C ::= \top \mid A \mid C \sqcap C \mid \exists r.C$, where $A \in \mathsf{N_C}$ and $r \in \mathsf{N_R}$. The set of all $\mathcal{EL}$-concepts is denoted with $\mathcal{EL}$.

The semantics is defined using interpretations $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$, where the domain $\Delta^\mathcal{I}$ is a non-empty set, and $\cdot^\mathcal{I}$ is a function mapping each concept name A to a subset $A^\mathcal{I}$ of $\Delta^\mathcal{I}$ and every role name r to a binary relation $r^\mathcal{I}$ over $\Delta^\mathcal{I}$. The *extension* $C^\mathcal{I}$ of a concept C is defined inductively as: $(\top)^\mathcal{I} := \Delta^\mathcal{I}$, $(C \sqcap D)^\mathcal{I} := C^\mathcal{I} \cap D^\mathcal{I}$, and $(\exists r.C)^\mathcal{I} := \{x \in \Delta^\mathcal{I} \mid \exists y \in C^\mathcal{I} : (x, y) \in r^\mathcal{I}\}$.

An *$\mathcal{EL}$-axiom* is either a *concept inclusion* $C \sqsubseteq D$, or a *concept equation* $C \equiv D$, for $\mathcal{EL}$-concepts C, D. An *$\mathcal{EL}$-TBox* $\mathcal{T}$ is a finite set of $\mathcal{EL}$-axioms. An interpretation $\mathcal{I}$ *satisfies* a concept inclusion $C \sqsubseteq D$ or a concept equation $C \equiv D$ if $C^\mathcal{I} \subseteq D^\mathcal{I}$ or $C^\mathcal{I} = D^\mathcal{I}$, respectively. An interpretation $\mathcal{I}$ *satisfies* a set Φ of $\mathcal{EL}$-axioms iff $\mathcal{I}$ satisfies all axioms in Φ; in this case, we say that $\mathcal{I}$ is a *model* of Φ. A set Φ_2 of $\mathcal{EL}$-axioms *follows* from a set of $\mathcal{EL}$-axioms Φ_1, written $\Phi_1 \models \Phi_2$, iff every model of Φ_1 is also a model of Φ_2. If $\Phi_2 = \{\alpha\}$, we simply write $\Phi_1 \models \alpha$ instead of $\Phi_1 \models \Phi_2$.

A signature Σ is a finite set of concept or role names from $\mathsf{N_C}$ and $\mathsf{N_R}$. The signature $\mathrm{sig}(\varphi)$ is the set of concept and role names occurring in φ, where φ ranges over any syntactic object. We set $\mathrm{sig}^{\mathsf{N_C}}(\varphi) := \mathrm{sig}(\varphi) \cap \mathsf{N_C}$. The symbol Σ is used as a subscript to a set of concepts or axioms to denote that the elements only use symbols from Σ, e.g., $\mathcal{EL}_\Sigma$, etc.

An $\mathcal{EL}$-TBox $\mathcal{T}$ is *normalised* iff it only consists of $\mathcal{EL}$-concept inclusions of the forms $\top \sqsubseteq B$, $A \sqsubseteq B$, $A_1 \sqcap A_2 \sqsubseteq B$, $A \sqsubseteq \exists r.B$, or $\exists r.A \sqsubseteq B$, where $A, A_1, A_2, B \in \mathsf{N_C}$ and $r \in \mathsf{N_R}$. Every $\mathcal{EL}$-TBox $\mathcal{T}$ can be normalised into a TBox $\mathrm{norm}(\mathcal{T})$ in polynomial time in the size of $\mathcal{T}$ with at most a linear increase in size such that (i) for every $C, D \in \mathcal{EL}$ with $\mathrm{sig}(C) \cup \mathrm{sig}(D) \subseteq \mathrm{sig}(\mathcal{T})$, it holds that $\mathrm{norm}(\mathcal{T}) \models C \sqsubseteq D$ iff $\mathcal{T} \models C \sqsubseteq D$ and (ii) for every $X \in \mathrm{sig}^{\mathsf{N_C}}(\mathrm{norm}(\mathcal{T})) \setminus \mathrm{sig}(\mathcal{T})$, there exists $D_X \in \mathcal{EL}$ with $\mathrm{sig}(D_X) \subseteq \mathrm{sig}(\mathcal{T})$ and $\mathrm{norm}(\mathcal{T}) \models X \equiv D_X$.

Given an $\mathcal{EL}$-concept C, the *depth of* C, denoted with $\mathrm{depth}(C)$, is inductively defined as follows: $\mathrm{depth}(\top) = \mathrm{depth}(A) = 0$ for $A \in \mathsf{N_C}$, $\mathrm{depth}(C_1 \sqcap C_2) = \max\{\mathrm{depth}(C_1), \mathrm{depth}(C_2)\}$, and $\mathrm{depth}(\exists r.C) = 1 + \mathrm{depth}(C)$.

We recall our graph representation of sets of concepts from [7]. A *concept set graph* is a finite, labelled, directed, graph $(\mathcal{V}, \mathcal{E}, \mathcal{L}, \rho)$ with a dedicated root node $\rho \in \mathcal{V}$, where $\mathcal{V}$ is a finite, non-empty set of nodes, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is a set of directed edges, $\mathcal{L} \colon \mathcal{V} \cup \mathcal{E} \to 2^{\mathsf{N_C}} \cup \mathsf{N_R}$ is a labelling function, mapping nodes $v \in \mathcal{V}$ to finite sets of concept names $\mathcal{L}(v) \subseteq \mathsf{N_C}$, and mapping edges $e \in \mathcal{E}$ to a role name $\mathcal{L}(e) \in \mathsf{N_R}$. We say that a concept set graph $\mathcal{G}' = (\mathcal{V}', \mathcal{E}', \mathcal{L}', \rho')$ is a *subgraph of a concept set graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{L}, \rho)$ iff $\mathcal{V}' \subseteq \mathcal{V}$, $\mathcal{L}'(v') \subseteq \mathcal{L}(v')$ for every $v' \in \mathcal{V}'$, and $\mathcal{L}'(e') = \mathcal{L}(e')$ for every $e' \in \mathcal{E}'$. A sequence $\pi = v_0 \cdot v_1 \cdot \ldots \cdot v_n$ with $n \geq 1$ and $(v_i, v_{i+1}) \in \mathcal{E}$ for every $0 \leq i \leq n-1$ is a *path in* $\mathcal{G}$. We write $v \in \pi$ iff $v = v_i$ for some $0 \leq i \leq n$. A path π is called a *cycle* if $v_0 = v_n$ and for every $i, j \in \{0, \ldots, n-1\}$ with $i \neq j$ it holds that $v_i \neq v_j$; π is called *acyclic* if $v_i \neq v_j$ for every $i, j \in \{0, \ldots, n\}$ with $i \neq j$.

$\mathcal{EL}$-concepts can be read off concept set graphs by unfolding the graph as follows. Let $\mathrm{Unfold}_\mathcal{G}(v, 0) := \bigsqcap_{A \in \mathcal{L}(v)} A$ and for $n > 0$, we set $\mathrm{Unfold}_\mathcal{G}(v, n) := \bigsqcap_{A \in \mathcal{L}(v)} A \sqcap \bigsqcap_{(v,w) \in \mathcal{E},\, \mathcal{L}(v,w) = r} \exists r.\mathrm{Unfold}_\mathcal{G}(w, n-1)$. Note that $\mathrm{Unfold}_\mathcal{G}(v, 0) = \top$ if $\mathcal{L}(v) = \emptyset$. Finally, let $\mathrm{Unfold}_\mathcal{G}(v) := \{\, \mathrm{Unfold}_\mathcal{G}(v, n) \mid n \geq 0 \,\}$ and $\mathrm{Unfold}(\mathcal{G}) := \mathrm{Unfold}_\mathcal{G}(\rho)$.

For a signature Σ, we define the *Σ-reduct of* $\mathcal{G}$, denoted with $\mathrm{reduct}_\Sigma(\mathcal{G})$, to be the subgraph $\mathrm{reduct}_\Sigma(\mathcal{G}) := (\mathcal{V}_\Sigma, \mathcal{E}_\Sigma, \mathcal{L}_\Sigma, \rho_\Sigma)$ of $\mathcal{G}$, where $\mathcal{V}_\Sigma := \mathcal{V}$, $\rho_\Sigma := \rho$, $\mathcal{E}_\Sigma := \{\, e \in \mathcal{E} \mid \mathcal{L}(e) \in \Sigma \,\}$, and $\mathcal{L}_\Sigma := \{\, (v, S) \mid v \in \mathcal{V},\, S = \mathcal{L}(v) \cap \Sigma \,\} \cup \{\, (e, S) \mid e \in \mathcal{E}_\Sigma,\, S = \mathcal{L}(e) \,\}$.

We will use simulations between concept set graphs $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1, \mathcal{L}_1, \rho_1)$ and $\mathcal{G}_2 = (\mathcal{V}_2, \mathcal{E}_2, \mathcal{L}_2, \rho_2)$ to determine whether every unfolding of $\mathcal{G}_1$ is entailed by an unfolding of $\mathcal{G}_2$ [4, 6, 17]. For $u_1 \in \mathcal{V}_1$ and $u_2 \in \mathcal{V}_2$, we say that a relation $S \subseteq \mathcal{V}_1 \times \mathcal{V}_2$ is a *subsumer simulation between* $\mathcal{G}_1$ *and* $\mathcal{G}_2$ iff the following conditions hold:

(i) if $(v_1, v_2) \in S$, then $\mathcal{L}_1(v_1) \subseteq \mathcal{L}_2(v_2)$; and

(ii) if $(v_1, v_2) \in S$, $e_1 = (v_1, v_1') \in \mathcal{E}_1$, and $r := \mathcal{L}_1(e_1)$, then there exists $e_2 = (v_2, v_2') \in \mathcal{E}_2$ such that $r = \mathcal{L}_2(e_2)$ and $(v_1', v_2') \in S$.

We say that u_2 *subsumer simulates* u_1, denoted with $\text{sim}_{\rightarrow}([\mathcal{G}_1, u_1], [\mathcal{G}_2, u_2])$, iff there exists a subsumer graph simulation S between $\mathcal{G}_1$ and $\mathcal{G}_2$ with $(u_1, u_2) \in S$. We write $\text{sim}_{\rightarrow}(\mathcal{G}_1, \mathcal{G}_2)$ iff $\text{sim}_{\rightarrow}([\mathcal{G}_1, \rho_1], [\mathcal{G}_2, \rho_2])$ holds. One can show that $\text{sim}_{\rightarrow}([\mathcal{G}_1, u_1], [\mathcal{G}_2, u_2])$ holds iff for every $C \in \text{Unfold}_{\mathcal{G}_1}(u_1)$ there exists $D \in \text{Unfold}_{\mathcal{G}_2}(u_2)$ with $\models D \sqsubseteq C$ [7].

3 Subsumer Interpolants

A uniform interpolant of a TBox $\mathcal{T}$ w.r.t. a signature Σ is an exact, finite representation in terms of a TBox of the (potentially) infinite set of Σ-consequences of $\mathcal{T}$ using symbols from Σ only. A more general notion takes a specific set Φ of Σ-consequences into account. Here we are interested in the set Φ of $\mathcal{EL}$-subsumers of a concept name w.r.t. an $\mathcal{EL}$-TBox, leading to the notion of Σ-subsumer interpolants.

Definition 1 (Σ-Uniform Interpolant)**.** *Let $\mathcal{T}$ be an $\mathcal{EL}$-TBox, let Σ be a signature, and let Φ be a set of $\mathcal{EL}_\Sigma$-concept inclusions. A TBox $\mathcal{T}_\Sigma$ is called* a (Σ, Φ)-uniform interpolant of $\mathcal{T}$ *iff the following conditions hold: (i) sig$(\mathcal{T}_\Sigma) \subseteq \Sigma$; (ii) $\mathcal{T} \models \mathcal{T}_\Sigma$; and (iii) for every $\alpha \in \Phi$ with $\mathcal{T} \models \alpha$ it holds that $\mathcal{T}_\Sigma \models \alpha$.*

A (Σ, Φ)-uniform interpolant $\mathcal{T}_\Sigma$ of $\mathcal{T}$ is called a Σ-uniform interpolant *if $\Phi = \{ C \sqsubseteq D \mid C, D \in \mathcal{EL}_\Sigma \}$; and a* Σ-subsumer interpolant of $A \in \Sigma$ w.r.t. $\mathcal{T}$ *if $\Phi = \{ A \sqsubseteq D \mid D \in \mathcal{EL}_\Sigma \}$.*

The following folklore example shows that Σ-subsumer interpolants do not always exist, in which case a Σ-uniform interpolants does not exist either.

Example 1. *Let $\mathcal{T} = \{A \sqsubseteq X,\ X \sqsubseteq \exists r.X\}$ and let $\Sigma = \{A, r\}$. Then a Σ-subsumer interpolant of A w.r.t. $\mathcal{T}$ does not exist as the infinite sequence of consequences $A \sqsubseteq \exists r.\top$, $A \sqsubseteq \exists r.\exists r.\top$, etc. cannot be captured in a (finite) TBox $\mathcal{T}_\Sigma$ with sig$(\mathcal{T}_\Sigma) \subseteq \Sigma$ and $\mathcal{T} \models \mathcal{T}_\Sigma$.*

The converse does not hold, i.e., the existence of Σ-subsumer interpolants does not imply the existence of a Σ-uniform interpolant as the following example shows.

Example 2. *Let $\mathcal{T} = \{B \sqsubseteq X,\ \exists r.X \sqsubseteq X,\ X \sqsubseteq A\}$ and let $\Sigma = \{A, B, r\}$. Then $\mathcal{T}_\Sigma^A = \emptyset$ and $\mathcal{T}_\Sigma^B = \{B \sqsubseteq A\}$ are Σ-subsumer interpolants of A and B w.r.t. $\mathcal{T}$, respectively. However, a Σ-uniform interpolant of $\mathcal{T}$ does not exist as the infinite sequence of consequences $\exists r.B \sqsubseteq A$, $\exists r.\exists r.B \sqsubseteq A$, etc. cannot be captured in a (finite) TBox $\mathcal{T}_\Sigma$ with sig$(\mathcal{T}_\Sigma) \subseteq \Sigma$ and $\mathcal{T} \models \mathcal{T}_\Sigma$.*

However, there are exceptions as for some $\mathcal{EL}$-TBoxes $\mathcal{T}$ all Σ-consequences already follow from $\mathcal{EL}_\Sigma$-concept inclusions of the form $A \sqsubseteq D$ entailed by $\mathcal{T}$. A prominent example of such a TBox is the ($\mathcal{EL}$-fragment of the) ontology ChEBI (release January 6, 2016).

Lemma 1. *Let $\mathcal{T}$ be a TBox whose normalisation consists of inclusions of the form $X \sqsubseteq Y$ and $X \sqsubseteq \exists r.Y$, and let Σ be a signature. Then a Σ-uniform interpolant of $\mathcal{T}$ exists iff for every $A \in \Sigma$, there exists a Σ-subsumer interpolant of A w.r.t. $\mathcal{T}$.*[1]

[1] The $\mathcal{EL}$-fragment mentioned in Lemma 1 is a fragment of OWL 2 QL and it is expressible in DL-Lite$^{\mathcal{H}}_{\text{core}}$. To the best of our knowledge, there are no results regarding uniform interpolation for OWL 2 QL and DL-Lite$^{\mathcal{H}}_{\text{core}}$. Existing results for DL-Lite either do not consider role inclusions or they focus on conjunctive queries as a query language [13, 26].

We show that the problem of deciding the existence of a Σ-uniform interpolant can be reduced to deciding the existence of (Σ, Ψ)-uniform interpolants for certain subsets Ψ of the set of all $\mathcal{EL}_\Sigma$-concept inclusions that follow from $\mathcal{T}$.

Theorem 1. *Let $\mathcal{T}$ be a normalised $\mathcal{EL}$-TBox, let Σ be a signature, and let $\Phi = \{ C \sqsubseteq D \mid C, D \in \mathcal{EL}_\Sigma, \mathcal{T} \models C \sqsubseteq D \}$. Additionally, let $\Psi_1, \ldots, \Psi_n$ with $n \geq 1$ be such that $\bigcup_{i=1}^n \Psi_i \models \Phi$ and $\Psi_i \subseteq \Phi$ for every $1 \leq i \leq n$.*

Then there exists a (Σ, Φ)-uniform interpolant of $\mathcal{T}$ iff for every $1 \leq i \leq n$, there exists a (Σ, Ψ_i)-uniform interpolant of $\mathcal{T}$.

We split the set Φ_Σ of all $\mathcal{EL}_\Sigma$-inclusions entailed by $\mathcal{T}$ into the following sets: for every $A \in \Sigma$, $\Phi_A^{\rightarrow} := \{ C \sqsubseteq D \in \Phi_\Sigma \mid C = A \}$ and $\Phi_A^{\leftarrow} := \{ C \sqsubseteq D \in \Phi_\Sigma \mid D = A \}$; for every $X \in \mathrm{sig}(\mathcal{T}) \setminus \Sigma$, $\Phi_X^{\mathrm{m}} := \{ C \sqsubseteq D \mid C, D \in \mathcal{EL}_\Sigma, \mathcal{T} \models C \sqsubseteq X \text{ and } \mathcal{T} \models X \sqsubseteq D \}$; and $\Phi_\top^{\rightarrow} := \{ C \sqsubseteq D \in \Phi_\Sigma \mid C = \top \}$. Similar to the proof of the so-called *Witness Theorem* for the Logical Difference of $\mathcal{EL}$-TBoxes (Theorem 7 in [7]), we can show that

$$\bigcup_{A \in \Sigma} (\Phi_A^{\rightarrow} \cup \Phi_A^{\leftarrow}) \cup \bigcup_{X \in \mathrm{sig}(\mathcal{T}) \setminus \Sigma} \Phi_X^{\mathrm{m}} \cup \Phi_\top^{\rightarrow} \models \Phi_\Sigma$$

by analysing the derivation of $\mathcal{T} \models \varphi$ for $\varphi \in \Phi_\Sigma$ in a Gentzen-style calculus for $\mathcal{EL}$-TBoxes that we have adapted from [9].

In this paper, we develop a technique for deciding the existence of subsumer interpolants. More precisely, given an $\mathcal{EL}$-TBox $\mathcal{T}$, a signature Σ, and a concept name $A \in \Sigma$, we want to decide whether there exists a Σ-subsumer interpolant of A w.r.t. $\mathcal{T}$. It is easy to see that a Σ-subsumer interpolant of $\mathcal{T}$ exists iff a Σ-subsumer interpolant of $\mathrm{norm}(\mathcal{T})$ exists. To simplify the notation, we concentrate on $\mathcal{EL}$-TBoxes $\mathcal{T}$ that are normalised.

In order to facilitate the construction of $\mathcal{T}_{\mathrm{desc}}^{\Sigma,A}$ and to verify that it captures all the Σ-subsumers of A w.r.t. $\mathcal{T}$, we represent its set of subsumers using dedicated concept set graphs, called *subsumer graphs*. Such subsumer graphs are defined using *logical reasoning*, for which we make use of our sequent calculus for $\mathcal{EL}$ [9] in our correctness and completeness proofs. Similar graph notions appeared in the literature [11,20].

Definition 2 (Subsumer Graph for a Concept Name w.r.t. a TBox)**.** *Let $\mathcal{T}$ be a normalised $\mathcal{EL}$-TBox, and let $A \in \Sigma$. The* subsumer graph *of A w.r.t. $\mathcal{T}$, denoted with $\exp_{\rightarrow}^{\mathcal{T}}(A)$, is a concept set graph $(\mathcal{V}, \mathcal{E}, \mathcal{L}, \rho)$, where $\mathcal{V} := \{v_A^\epsilon\} \cup \{ v_\varphi^r \mid X \sqsubseteq \exists r.\varphi \in \mathcal{T} \}$, $\rho := v_A^\epsilon$,*

$$\mathcal{E} := \{ (v_\varphi^\lambda, v_\psi^r) \mid X \sqsubseteq \exists r.\psi \in \mathcal{T}, \mathcal{T} \models \varphi \sqsubseteq X \}, \text{ and}$$
$$\mathcal{L} := \{ (v_\varphi^\lambda, S) \mid v_\varphi^\lambda \in \mathcal{V}, S = \{ Y \in sig^{\mathsf{N_C}}(\mathcal{T}) \mid \mathcal{T} \models \varphi \sqsubseteq Y \} \} \cup \{ (e, r) \mid e = (v_\varphi^\lambda, v_\psi^r) \}.$$

For a node $v_\varphi^\lambda \in \mathcal{V}$, the concept name $\varphi \in sig^{\mathsf{N_C}}(\mathcal{T}) \cup \{A\}$, denoted with $gen_{\mathcal{G}}(v_\varphi^\lambda)$, is called the generating concept name *of v_φ^λ.*

The root node of a subsumer graph $\exp_{\rightarrow}^{\mathcal{T}}(A)$ is denoted with v_A^ϵ. For every node v_φ^λ with $\lambda \in \mathsf{N_R} \cup \{\epsilon\}$ and for every axiom $X \sqsubseteq \exists r.\psi \in \mathcal{T}$ with $\mathcal{T} \models \varphi \sqsubseteq X$ there is an edge $(v_\varphi^\lambda, v_\psi^r)$ labelled with r to a successor node v_ψ^r. All nodes v_ψ^r, except the root node, represent the right-hand side of an axiom of the form $X \sqsubseteq \exists r.\psi$. The concept name ψ is the generating concept name $\mathrm{gen}_{\mathcal{G}}(v_\psi^r)$ of v_ψ^r. A leaf node v with $\mathcal{L}(v) = \emptyset$ represents the $\top$-concept.

For every node v in $\mathcal{G} = \exp_{\rightarrow}^{\mathcal{T}}(A)$, one can show that the set of subsumers of the concept name $\mathrm{gen}_{\mathcal{G}}(v)$ w.r.t. $\mathcal{T}$ are equivalent to the set $\mathrm{Unfold}_{\mathcal{G}}(v)$. Additionally, by taking the Σ-

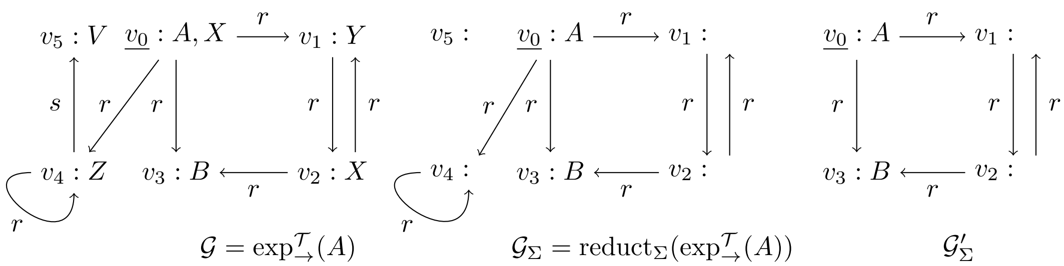

Figure 1: Subsumer graph and corresponding Σ-reduct for Example 3

reduct $\mathcal{G}_\Sigma \coloneqq \text{reduct}_\Sigma(\mathcal{G})$ of $\mathcal{G}$ we have that the set of $\mathcal{EL}_\Sigma$-subsumers of A w.r.t. $\mathcal{T}$ is equivalent to the set $\text{Unfold}_{\mathcal{G}_\Sigma}(v)$.

Example 3. *Let $\mathcal{T}$ be the following normalised $\mathcal{EL}$-TBox:*

$$\begin{array}{llll} A \sqsubseteq X & X \sqsubseteq \exists r.B & \exists r.B \sqsubseteq X & Z \sqsubseteq \exists r.Z \\ A \sqsubseteq \exists r.Z & X \sqsubseteq \exists r.Y & Y \sqsubseteq \exists r.X & Z \sqsubseteq \exists s.V \end{array}$$

For $\Sigma \coloneqq \{A, B, r\}$, it holds that $\mathcal{T}_\Sigma \coloneqq \{A \sqsubseteq \exists r.B, \exists r.B \sqsubseteq \exists r.\exists r.\exists r.B\}$ is a Σ-subsumer interpolant of A w.r.t. $\mathcal{T}$. However, for $\Sigma' \coloneqq \{A, B, r, s\}$, there does not exist a Σ'-subsumer interpolant of A w.r.t. $\mathcal{T}$.

The subsumer graph $\mathcal{G} = exp^{\mathcal{T}}_{\rightarrow}(A)$ and the concept set graph $\mathcal{G}_\Sigma = reduct_\Sigma(exp^{\mathcal{T}}_{\rightarrow}(A))$ are depicted in Figure 1. Note that in depictions of concept set graphs $(\mathcal{V}, \mathcal{E}, \mathcal{L}, \rho)$ in this paper, nodes $v \in \mathcal{V}$ are represented together with their labels in the form '$v : \mathcal{L}(v)$', and the root node ρ is underlined. Edges and their labels are represented as labelled arrows between the nodes.

We have $gen_{\mathcal{G}}(v_0) = A$, $gen_{\mathcal{G}}(v_1) = Y$, $gen_{\mathcal{G}}(v_2) = X$, $gen_{\mathcal{G}}(v_3) = B$, $gen_{\mathcal{G}}(v_4) = Z$, and $gen_{\mathcal{G}}(v_5) = V$. It holds that $Unfold_{\mathcal{G}_\Sigma}(\rho, 0) = A$ and $Unfold_{\mathcal{G}_\Sigma}(\rho, 1) = A \sqcap \exists r.\top \sqcap \exists r.B \sqcap \exists r.\top$.

In case the Σ-reduct $\mathcal{G}_\Sigma$ of the subsumer graph $\mathcal{G} = \exp^{\mathcal{T}}_{\rightarrow}(A)$ is acyclic, there only exist finitely many non-equivalent Σ-subsumers of A w.r.t. $\mathcal{T}$. In particular, there exists a number $n \geq 0$ such that for every $m \geq n$ it holds that $\text{Unfold}_{\mathcal{G}_\Sigma}(\rho, n) = \text{Unfold}_{\mathcal{G}_\Sigma}(\rho, m)$. Then $\{A \sqsubseteq \text{Unfold}_{\mathcal{G}_\Sigma}(\rho, n)\}$ is a Σ-subsumer interpolant of A w.r.t. $\mathcal{T}$. In general, subsumer interpolants of $\mathcal{EL}$-TBoxes whose Σ-reduct of the subsumer graph of a concept name is acyclic always exist.

Example 4. *Let $\mathcal{T} \coloneqq \{A \sqsubseteq X, X \sqsubseteq B, X \sqsubseteq \exists r.Y, Y \sqsubseteq B, A \sqsubseteq \exists r.Z, Z \sqsubseteq \exists s.Z\}$, and let $\Sigma \coloneqq \{A, B, r\}$. Let $\mathcal{G} \coloneqq exp^{\mathcal{T}}_{\rightarrow}(A)$ with root ρ, and let $\mathcal{G}_\Sigma = reduct_\Sigma(\mathcal{G})$. Observe that $\mathcal{G}$ is cyclic, but, as the role name s is not in Σ, the Σ-reduct $\mathcal{G}_\Sigma$ of $\mathcal{G}$ is acyclic. It holds that $C \coloneqq Unfold_{\mathcal{G}_\Sigma}(\rho, 1) = A \sqcap B \sqcap \exists r.B \sqcap \exists r.\top$, and $C = Unfold_{\mathcal{G}_\Sigma}(\rho, m)$ for every $m \geq 1$. Moreover, it holds that $\{A \sqsubseteq C\}$ is a Σ-subsumer interpolant of A w.r.t. $\mathcal{T}$.*

Handling cycles in $\mathcal{G}_\Sigma$ constitutes the main difficulty for deciding the existence of subsumer interpolants. For instance, the TBox $\mathcal{T}$ and the signature Σ in Example 1 give rise to a subsumer graph whose Σ-reduct $\mathcal{G}_\Sigma$ contains a cycle that cannot be described in terms of $\mathcal{EL}_\Sigma$-concept inclusions that follow from $\mathcal{T}$. Another example is the TBox from Example 4 together with the signature $\Sigma' = \{A, B, r, s\}$. However, there are cycles that can be described using $\mathcal{EL}_\Sigma$-concept inclusions that follow from $\mathcal{T}$. More precisely, a cycle $\pi = v_0 \cdot v_1 \cdot \ldots \cdot v_n$ in $\mathcal{G}_\Sigma$ is called Σ-*describable w.r.t.* $\mathcal{T}$ iff there exists an edge (v_i, v_{i+1}) with $1 \leq i < n$ such that there exists an $\mathcal{EL}_\Sigma$-concept C with $\mathcal{T} \models \text{gen}_{\mathcal{G}}(v_i) \sqsubseteq C$ and $\mathcal{T} \models C \sqsubseteq X$, where $X \sqsubseteq \exists r.Y \in \mathcal{T}$ such that $Y = \text{gen}_{\mathcal{G}}(v_{i+1})$ and $\mathcal{L}(v_i, v_{i+1}) = r$.

Furthermore, there are $\mathcal{EL}$-TBoxes $\mathcal{T}$ that together with a concept name A give rise to subsumer graphs $\mathcal{G} = \exp^{\mathcal{T}}_{\rightarrow}(A)$ whose Σ-reducts $\mathcal{G}_\Sigma$ contain both, cycles that are Σ-describable and cycles that are not Σ-describable. However, a Σ-subsumer interpolant of A w.r.t. $\mathcal{T}$ may exist nevertheless as it is illustrated in Example 3. We can determine whether a Σ-subsumer interpolant of A w.r.t. $\mathcal{T}$ exists in two steps. First, we construct a concept set graph $\mathcal{G}^{\Sigma,A}_{desc}$ that is essentially the fragment of $\mathcal{G}_\Sigma$ that does not contain any cycles that are not Σ-describable. Second, we determine whether $\mathrm{sim}_{\rightarrow}(\mathcal{G}_\Sigma, \mathcal{G}^{\Sigma,A}_{desc})$ holds by checking for the existence of a subsumer simulation between $\mathcal{G}_\Sigma$ and $\mathcal{G}^{\Sigma,A}_{desc}$.

Example 5. *Let $\mathcal{T}$ and Σ be as in Example 3. Consider the Σ-reduct $\mathcal{G}_\Sigma$ of the subsumer graph $\mathcal{G} = exp^{\mathcal{T}}_{\rightarrow}(A)$ depicted in Figure 1. Let $\pi_1 = v_2 \cdot v_1 \cdot v_2$ and $\pi_2 = v_4 \cdot v_4$ be two paths in $\mathcal{G}_\Sigma$. The path π_1 is produced by the axioms $\{X \sqsubseteq \exists r.B,\ \exists r.B \sqsubseteq X,\ X \sqsubseteq \exists r.Y,\ Y \sqsubseteq \exists r.X\} \subseteq \mathcal{T}$, and π_2 is produced by the axiom $Z \sqsubseteq \exists r.Z \in \mathcal{T}$. Both paths are cycles: π_1 is a Σ-describable cycle w.r.t. $\mathcal{T}$, whereas $\pi_2 = v_4 \cdot v_4$ is not.*

To see that π_1 is Σ-describable, note that for the edge (v_2, v_1) in $\mathcal{G}_\Sigma$ it holds that $\mathcal{T} \models X \sqsubseteq C$, and $\mathcal{T} \models C \sqsubseteq X$ for $C = \exists r.B$ with $X \sqsubseteq \exists r.Y \in \mathcal{T}$, $gen_{\mathcal{G}_\Sigma}(v_1) = Y$, and $gen_{\mathcal{G}_\Sigma}(v_2) = X$. We can build a Σ-concept inclusion α that describes π_1 and that follows from $\mathcal{T}$ by traversing π_1, starting from v_2. We set C to be the left-hand side of α, and we iteratively construct its right-hand side by describing the nodes and edges in π_1. We obtain $\alpha = \exists r.B \sqsubseteq \exists r.\exists r.\exists r.B$.

It holds that $sim_{\rightarrow}([\mathcal{G}_\Sigma, v_4], [\mathcal{G}_\Sigma, v_1])$, which implies that for every $C \in Unfold_{\mathcal{G}_\Sigma}(v_4)$ there exists $D \in Unfold_{\mathcal{G}_\Sigma}(v_1)$ with $\models D \sqsubseteq C$.

In order to obtain $\mathcal{G}^{\Sigma,A}_{desc}$, we first construct a TBox $\mathcal{T}^{\Sigma,A}_{\mathrm{desc}}$ that contains a description of all the Σ-describable cycles in $\mathcal{T}$. $\mathcal{G}^{\Sigma,A}_{desc}$ is then the subsumer graph of A w.r.t. $\mathcal{T}^{\Sigma,A}_{desc}$.

We present a procedure for finding Σ-describable cycles in $\mathcal{G}_\Sigma$ in Section 3.1. To decide whether a Σ-subsumer interpolant of A w.r.t. $\mathcal{T}$ exists, it is then sufficient to determine whether $\mathcal{G}^{\Sigma,A}_{desc}$ subsumer simulates $\mathcal{G}_\Sigma$. The check for the existence of such a simulation corresponds to verifying whether the unfoldings of every non-Σ describable cycle in $\mathcal{T}$ are covered by a Σ-describable cycle. The construction of $\mathcal{T}^{\Sigma,A}_{\mathrm{desc}}$ together with the decision procedure for subsumer interpolants is presented in Section 3.2.

3.1 Identifying Σ-Describable Cycles

We have to find all the edges $(v, w) \in \mathcal{E}$ in $\mathcal{G}_\Sigma$ for which there exists a concept $C \in \mathcal{EL}_\Sigma$ such that $\mathcal{T} \models \mathrm{gen}_{\mathcal{G}}(v) \sqsubseteq C$ and $\mathcal{T} \models C \sqsubseteq X$, where an axiom $X \sqsubseteq \exists r.Y$ occurs in $\mathcal{T}$ with $\mathcal{L}(v, w) = r$ and $Y = \mathrm{gen}_{\mathcal{G}}(w)$. To that end we will search for all pairs of concept names (X, Y) occurring in $\mathcal{T}$ for which there exists $C \in \mathcal{EL}_\Sigma$ with $\mathcal{T} \models X \sqsubseteq C$ and $\mathcal{T} \models C \sqsubseteq Y$. The concept C can be seen to be a Σ-interpolation concept between X and Y w.r.t. $\mathcal{T}$.

In the following we introduce a mapping $\mathrm{desc}^{\Sigma}_{\mathcal{T}}$, called the Σ-prime interpolation relation of $\mathcal{T}$, that assigns pairs (X, Y) of concept names occurring in $\mathcal{T}$ an $\mathcal{EL}$-concept $C \coloneqq \mathrm{desc}^{\Sigma}_{\mathcal{T}}(X, Y)$ iff there exists an $\mathcal{EL}_\Sigma$-concept D with $\mathcal{T} \models X \sqsubseteq D$ and $\mathcal{T} \models D \sqsubseteq Y$. We call such concepts D *Σ-prime interpolants of X and Y w.r.t. $\mathcal{T}$*. Additionally, we construct a TBox $\mathcal{T}^{\Sigma}_{\rightleftharpoons}$ from which one can build up a suitable prime interpolant D by unfolding the concept C, given the definitions in $\mathcal{T}^{\Sigma}_{\rightleftharpoons}$. Intuitively, the TBox $\mathcal{T}^{\Sigma}_{\rightleftharpoons}$ allows us to represent the concept D using structure sharing, which helps to improve the performance of our decision procedure in practice. The formal definition of the mapping $\mathrm{desc}^{\Sigma}_{\mathcal{T}}$ is as follows.

Definition 3 (Prime Interpolants)**.** *Let $\mathcal{T}$ be a normalised $\mathcal{EL}$-TBox and let Σ be a signature. For every $X, Y \in sig^{\mathsf{N_C}}(\mathcal{T})$, let $N_{X,Y} \in \mathsf{N_C} \setminus (sig(\mathcal{T}) \cup \Sigma)$ be a fresh concept name. We define*

$desc^{\Sigma}_{\mathcal{T}} \subseteq (sig^{\mathsf{N_C}}(\mathcal{T}) \times sig^{\mathsf{N_C}}(\mathcal{T})) \times \mathcal{EL}$ to be a Σ-prime interpolation relation of $\mathcal{T}$ *as* $desc^{\Sigma}_{\mathcal{T}} \coloneqq \bigcup_{i\geq 0} \Phi^{i}_{\mathcal{T},\Sigma}$, *where*

$$\Phi^{0}_{\mathcal{T},\Sigma} \coloneqq \{\, ((A,A),A) \mid A \in \Sigma \,\} \cup \{\, ((X,Y),\top) \mid X \in sig^{\mathsf{N_C}}(\mathcal{T}), \top \sqsubseteq Y \in \mathcal{T} \,\},$$

and for $i > 0$, $\Phi^{i}_{\mathcal{T},\Sigma}$, is a smallest set closed under the following conditions:

(i) $\Phi^{i-1}_{\mathcal{T},\Sigma} \subseteq \Phi^{i}_{\mathcal{T},\Sigma}$;

(ii) *if* $(X,Y) \in dom(\Phi^{i-1}_{\mathcal{T},\Sigma})$, $X',Y' \in sig^{\mathsf{N_C}}(\mathcal{T})$, $\mathcal{T} \models X' \sqsubseteq X$, $\mathcal{T} \models Y \sqsubseteq Y'$, *and* $(X',Y') \notin dom(\Phi^{i-1}_{\mathcal{T},\Sigma})$, *then* $((X',Y'),N_{X,Y}) \in \Phi^{i}_{\mathcal{T},\Sigma}$;

(iii) *if* $(X,Y_1),(X,Y_2) \in dom(\Phi^{i-1}_{\mathcal{T},\Sigma})$, $Y_1 \sqcap Y_2 \sqsubseteq Z \in \mathcal{T}$, *and* $(X,Z) \notin dom(\Phi^{i-1}_{\mathcal{T},\Sigma})$, *then* $((X,Z),N_{X,Y_1} \sqcap N_{X,Y_2}) \in \Phi^{i}_{\mathcal{T},\Sigma}$; *and*

(iv) *if* $(Y_1,Y_2) \in dom(\Phi^{i-1}_{\mathcal{T},\Sigma})$ *and there exist* $X \sqsubseteq \exists r.Y_1 \in \mathcal{T}$, $\exists r.Y_2 \sqsubseteq Z \in \mathcal{T}$ *with* $r \in \Sigma$, *and* $(X,Z) \notin dom(\Phi^{i-1}_{\mathcal{T},\Sigma})$, *then* $((X,Z),\exists r.N_{Y_1,Y_2}) \in \Phi^{i}_{\mathcal{T},\Sigma}$.

Finally, we set

$$\mathcal{T}^{\Sigma}_{\rightleftharpoons} \coloneqq \bigcup_{((X,Y),C)\in desc^{\Sigma}_{\mathcal{T}}} \{N_{X,Y} \equiv C\}.$$

The definition of $\mathrm{desc}^{\Sigma}_{\mathcal{T}}$ is similar to the inference rules present in consequence-based calculi. Obviously, all the concept names $A \in \Sigma$ have prime interpolants and they are handled in the set $\Phi^{0}_{\mathcal{T},\Sigma}$. Then, if we have already found a prime interpolant for a pair (X,Y) and there exist concept names X',Y' with $\mathcal{T} \models X' \sqsubseteq X$ and $\mathcal{T} \models Y \sqsubseteq Y'$, we can immediately derive that the pair (X',Y') has the same prime interpolant, which is handled by Condition *(ii)*. Condition *(iii)* takes axioms of the form $Y_1 \sqcap Y_2 \sqsubseteq Z$ into account. If we know that (X,Y_1) and (X,Y_2) have prime interpolants, then also the pair (X,Z) has a prime interpolant. Condition *(iv)* handles subsumption via existentials, involving axioms of the form $X \sqsubseteq \exists r.Y_1$ and $\exists r.Y_2 \sqsubseteq Z$ for which it is known already that (Y_1,Y_2) has a prime interpolant. Finally, the set $\Phi^{0}_{\mathcal{T},\Sigma}$ is also responsible for handling axioms of the form $\top \sqsubseteq X$. Note that the relation $\mathrm{desc}^{\Sigma}_{\mathcal{T}}$ is functional and that $(X,Y) \in \mathrm{dom}(\mathrm{desc}^{\Sigma}_{\mathcal{T}})$ implies $\mathcal{T} \models X \sqsubseteq Y$.

Example 6. *Let* $\mathcal{T} = \{X \sqsubseteq A,\ A \sqsubseteq Y,\ X \sqsubseteq \exists r.Z_1,\ \exists r.Z_2 \sqsubseteq Y,\ Z_1 \sqsubseteq B_1,\ Z_1 \sqsubseteq B_2,\ B_1 \sqcap B_2 \sqsubseteq Z_2\}$ *and let* $\Sigma = \{A, B_1, B_2, r\}$. *Then we have that*

$$\begin{aligned} desc^{\Sigma}_{\mathcal{T}} \coloneqq \{&((A,A),A),\ ((B_1,B_1),B_1),\ ((B_2,B_2),B_2),\ ((X,A),N_{A,A}),\ ((A,Y),N_{A,A}),\\ &((Z_1,B_1),N_{B_1,B_1}),\ ((Z_1,B_2),N_{B_2,B_2}),\ ((Z_1,Z_2),N_{Z_1,B_1} \sqcap N_{Z_1,B_2}),\ ((X,Y),\exists r.N_{Z_1,Z_2})\} \end{aligned}$$

is a Σ-prime interpolation relation of $\mathcal{T}$. It holds that $\mathcal{T}^{\Sigma}_{\rightleftharpoons} \models N_{X,A} \equiv A$, $\mathcal{T}^{\Sigma}_{\rightleftharpoons} \models N_{Z_1,Z_2} \equiv B_1 \sqcap B_2$, *and* $\mathcal{T}^{\Sigma}_{\rightleftharpoons} \models N_{X,Y} \equiv \exists r.(B_1 \sqcap B_2)$. *Furthermore, we have* $\mathcal{T} \models X \sqsubseteq A$, $\mathcal{T} \models A \sqsubseteq Y$; $\mathcal{T} \models Z_1 \sqsubseteq B_1 \sqcap B_2$, $\mathcal{T} \models B_1 \sqcap B_2 \sqsubseteq Z_2$; *and* $\mathcal{T} \models X \sqsubseteq \exists r.(B_1 \sqcap B_2)$, $\mathcal{T} \models \exists r.(B_1 \sqcap B_2) \sqsubseteq Y$.

Note that for a given TBox $\mathcal{T}$ and a signature Σ, the Σ-prime interpolation relation is not unique. However, for our purposes it is sufficient to choose *one* prime interpolation relation, i.e. it is not important which interpolation concept $D \in \mathcal{EL}_{\Sigma}$ is selected for a prime interpolation pair (X,Y) - any will suffice.

We obtain the following correctness and completeness results for our computation procedure of prime interpolation relations.

Lemma 2. *Let $\mathcal{T}$ be a normalised $\mathcal{EL}$-TBox and let Σ be a signature. Then the following properties hold:*

(i) for every $((X,Y),C) \in desc^{\Sigma}_{\mathcal{T}}$ there exists $D \in \mathcal{EL}_\Sigma$ with $\mathcal{T}^{\Sigma}_{\rightleftharpoons} \models C \equiv D$, $\mathcal{T} \models X \sqsubseteq D$ and $\mathcal{T} \models D \sqsubseteq Y$;

(ii) if there exists $D \in \mathcal{EL}_\Sigma$ with $\mathcal{T} \models X \sqsubseteq D$ and $\mathcal{T} \models D \sqsubseteq Y$, then $(X,Y) \in dom(desc^{\Sigma}_{\mathcal{T}})$;

(iii) for every $N_{X,Y} \in sig(\mathcal{T}^{\Sigma}_{\rightleftharpoons})$ there exists $C \in \mathcal{EL}_\Sigma$ with $\mathcal{T}^{\Sigma}_{\rightleftharpoons} \models N_{X,Y} \equiv C$;

(iv) the TBox $\mathcal{T}^{\Sigma}_{\rightleftharpoons}$ can be computed in polynomial time in the size of $\mathcal{T}$.

The proof of Lemma 2 is based on analysing the derivations produced by our sequent calculus for characterising subsumption w.r.t. $\mathcal{EL}$-TBoxes.

Finally, we introduce the set $\mathcal{E}^{\Sigma}_{\text{desc}}$ as the set of all edges (v,w) in $\mathcal{G}_\Sigma = \text{reduct}_\Sigma(\exp^{\mathcal{T}}_{\rightarrow}(A)) = (\mathcal{V},\mathcal{E},\mathcal{L},\rho)$ (where $\mathcal{T}$ is a normalised $\mathcal{EL}$-TBox and Σ is a signature) for which there exists a prime interpolant of $\text{gen}_{\mathcal{G}_\Sigma}(v)$ and X, where $X \sqsubseteq \exists r.\text{gen}_{\mathcal{G}_\Sigma}(w) \in \mathcal{T}$ and $\mathcal{L}(v,w) = r$. Such edges will be called Σ-*prime edges in* $\mathcal{G}_\Sigma$. Formally, given a Σ-prime interpolation relation $\text{desc}^{\Sigma}_{\mathcal{T}}$ of $\mathcal{T}$, we set

$$\mathcal{E}^{\Sigma}_{\text{desc}} \coloneqq \{\, (v,w) \in \mathcal{E} \mid \exists\, X \sqsubseteq \exists r.Y \in \mathcal{T},\ \text{gen}_{\mathcal{G}_\Sigma}(w) = Y,\ (\text{gen}_{\mathcal{G}_\Sigma}(v), X) \in \text{dom}(\text{desc}^{\Sigma}_{\mathcal{T}}) \,\}.$$

Moreover, we define a function $\text{desc}^{\Sigma}_{\mathcal{G}_\Sigma} \colon \mathcal{E}^{\Sigma}_{\text{desc}} \to \mathsf{N_C}$ mapping $(v,w) \in \mathcal{E}^{\Sigma}_{\text{desc}}$ to $\text{desc}^{\Sigma}_{\mathcal{G}_\Sigma}(v,w) \coloneqq N_{X,Y}$, where $X = \text{gen}_{\mathcal{G}_\Sigma}(v)$, $\text{gen}_{\mathcal{G}_\Sigma}(w) = Z$, and $(X,Y) \in \text{dom}(\text{desc}^{\Sigma}_{\mathcal{T}})$ for some $Y \sqsubseteq \exists r.Z \in \mathcal{T}$ with $\mathcal{L}(v,w) = r$.

3.2 Decision Procedure for Subsumer Interpolants

We now give a formal definition of the TBox $\mathcal{T}^{\Sigma,A}_{\text{desc}}$ that describes as much as possible of the structure of $\mathcal{G}_\Sigma = \text{reduct}_\Sigma(\exp^{\mathcal{T}}_{\rightarrow}(A))$ using Σ-consequences of $\mathcal{T}$ only.

Definition 4 (Maximal Σ-Description)**.** *Let $\mathcal{T}$ be a normalised $\mathcal{EL}$-TBox, let Σ be a signature, and let $A \in \Sigma$. Additionally, let $\mathcal{G}_\Sigma \coloneqq reduct_\Sigma(exp^{\rightarrow}_{\mathcal{T}}(A)) = (\mathcal{V},\mathcal{E},\mathcal{L},\rho)$ and let $desc^{\Sigma}_{\mathcal{T}}$ be a Σ-prime interpolation relation of $\mathcal{T}$. Moreover, for $v_0 \in \mathcal{V}$, let $\mathcal{E}_{ac}(v_0) \subseteq \mathcal{V}^*$ be the smallest set of sequences over $\mathcal{V}$ defined inductively as follows:*

(i) $v_0 \in \mathcal{E}_{ac}(v_0)$;

(ii) if $\pi \cdot v \in \mathcal{E}_{ac}(v_0)$, $(v,w) \in \mathcal{E}$, $w \notin \pi \cdot v$, $(v,w) \notin \mathcal{E}^{\Sigma}_{desc}$, then $\pi \cdot v \cdot w \in \mathcal{E}_{ac}(v_0)$.

Finally, we assume that for every $\pi \in \mathcal{V}^*$, $N_\pi \in \mathsf{N_C}$ is a fresh concept name, and we set:

$$\begin{aligned}\mathcal{T}^{\Sigma,A}_{desc} \coloneqq\ & \mathcal{T}^{\Sigma}_{\rightleftharpoons} \cup \{A \sqsubseteq N_\rho\}\\ & \cup \{\, desc^{\Sigma}_{\mathcal{G}_\Sigma}(v,w) \sqsubseteq \exists r.N_w \mid (v,w) \in \mathcal{E}^{\Sigma}_{desc}, \mathcal{L}(v,w) = r \,\}\\ & \cup \{\, N_{\pi\cdot v} \equiv \bigsqcap_{B \in \mathcal{L}(v)} B \sqcap \bigsqcap_{\substack{\pi\cdot v\cdot w \in \Pi_\Sigma\\ \mathcal{L}(v,w)=r}} \exists r.N_{\pi\cdot v\cdot w} \sqcap \bigsqcap_{(v,w)\in\mathcal{E}^{\Sigma}_{desc}} desc^{\Sigma}_{\mathcal{G}_\Sigma}(v,w) \mid \pi \cdot v \in \Pi_\Sigma \,\},\end{aligned}$$

where $\Pi_\Sigma = \mathcal{E}_{ac}(\rho) \cup \bigcup_{(v,w)\in\mathcal{E}^{\Sigma}_{desc}} \mathcal{E}_{ac}(w)$.

Recall that $\mathcal{E}^{\Sigma}_{\text{desc}}$ contains all the Σ-prime edges $(v,w) \in \mathcal{E}$ of $\mathcal{G}_\Sigma$ and that $\text{desc}^{\Sigma}_{\mathcal{G}_\Sigma}(v,w)$ is a concept name of the form $N_{X,Y} \in \text{sig}(\mathcal{T}^{\Sigma}_{\rightleftharpoons}) \setminus \Sigma$ corresponding to a concept $D \in \mathcal{EL}_\Sigma$ such that $\mathcal{T} \models \text{gen}_{\mathcal{G}_\Sigma}(v) \sqsubseteq D$ and $\mathcal{T} \models D \sqsubseteq \exists r.\text{gen}_{\mathcal{G}_\Sigma}(v)$ with $\mathcal{L}(v,w) = r$.

Intuitively, for a node $v \in \mathcal{V}$, the set $\mathcal{E}_{\text{ac}}(v)$ contains all acyclic paths π in $\mathcal{G}_\Sigma$ that start at v and that lead to a leaf node without traversing Σ-prime edges. The set Π_Σ contains all acyclic paths starting at the root node ρ or the target nodes of Σ-prime edges. For such paths $\pi = \pi' \cdot v$, the fresh concept name N_π describes the Σ-structure of the node v in $\mathcal{G}_\Sigma$: the Σ-concept names that are entailed by $\text{gen}_{\mathcal{G}_\Sigma}(v)$ and its implied Σ-edges, which can either be Σ-prime edges (v, w) that are described by the $\text{desc}^{\Sigma}_{\mathcal{G}_\Sigma}(v, w)$ concept name, or regular edges which are axiomatised using the $N_{\pi' \cdot v \cdot w}$ concept name. The set Π_Σ is of exponential size in $\mathcal{V}$ as it enumerates all the acyclic paths in $\mathcal{G}_\Sigma$. Note that all the role names occurring in $\mathcal{T}^{\Sigma,A}_{\text{desc}}$ are in Σ, which is not the case for the concept names.

We obtain the following properties of $\mathcal{T}^{\Sigma,A}_{\text{desc}}$ and of its associated subsumer graph $\mathcal{G}^{\Sigma,A}_{\text{desc}}$. Note that $\mathcal{T}^{\Sigma,A}_{\text{desc}}$ needs to be normalised first.

Lemma 3. *Let $\mathcal{T}$ be a normalised $\mathcal{EL}$-TBox, let Σ be a signature, and let $A \in \Sigma$. Additionally, let $\mathcal{T}' \coloneqq norm(\mathcal{T}^{\Sigma,A}_{desc})$ and let $\mathcal{G}^{\Sigma,A}_{desc} = exp^{\mathcal{T}'}_{\to}(A) = (\mathcal{V}, \mathcal{E}, \mathcal{L}, \rho)$. Then*

(i) for every $C, D \in \mathcal{EL}_\Sigma$ with $\mathcal{T}^{\Sigma,A}_{desc} \models C \sqsubseteq D$, it holds that $\mathcal{T} \models C \sqsubseteq D$;

(ii) for every $X \in sig^{\mathsf{N_C}}(\mathcal{T}')$, there exists $C \in \mathcal{EL}_\Sigma$ with $\mathcal{T}' \models X \equiv C$; and

(iii) $\mathcal{E}^{\Sigma}_{desc} = \mathcal{E}$.

Property (i) holds by construction of $\mathcal{T}^{\Sigma,A}_{\text{desc}}$. Using the properties of the TBox $\mathcal{T}^{\Sigma}_{\rightleftharpoons}$, it is easy to see that Property (ii) holds for all concept names of the form $N_{X,Y} \in \text{sig}(\mathcal{T}^{\Sigma}_{\rightleftharpoons})$. For concept names of the form N_π the property can be shown by induction on the length of π. Property (iii) follows from the construction of $\mathcal{E}_{\text{ac}}(v)$ for $v \in \Pi_\Sigma$.

Example 7. *Let $\mathcal{T}$ and let Σ be defined as in Example 3 and let $\mathcal{G}_\Sigma$ be as shown in Figure 1. Then we have $desc^{\Sigma}_{\mathcal{T}} = \{((A, A), A), ((B, B), B), ((A, X), N_{A,A}), ((X, X), \exists r.N_{B,B})\}$ and $\mathcal{E}^{\Sigma}_{desc} = \{(v_0, v_1), (v_0, v_3), (v_0, v_4), (v_2, v_1), (v_2, v_3)\}$. Additionally, $\mathcal{E}_{ac}(v_0) = \{v_0\}$, $\mathcal{E}_{ac}(v_1) = \{v_1, v_1 \cdot v_2\}$, $\mathcal{E}_{ac}(v_3) = \{v_3\}$, and $\mathcal{E}_{ac}(v_4) = \{v_4\}$. Finally, we obtain $\mathcal{T}^{\Sigma,A}_{desc} = \mathcal{T}_{\rightleftharpoons} \cup \{A \sqsubseteq N_{v_0}\} \cup \mathcal{T}_{\mathcal{E}^{\Sigma}_{desc}} \cup \mathcal{T}_{\Pi_\Sigma}$, where*

$$\begin{aligned}
\mathcal{T}_{\rightleftharpoons} &\coloneqq \{N_{A,A} \equiv A,\ N_{B,B} \equiv B,\ N_{A,X} \equiv N_{A,A},\ N_{X,X} \equiv \exists r.N_{B,B}\},\\
\mathcal{T}_{\mathcal{E}^{\Sigma}_{desc}} &\coloneqq \{N_{A,X} \sqsubseteq \exists r.N_{v_1},\ N_{A,X} \sqsubseteq \exists r.N_{v_3},\ N_{A,A} \sqsubseteq \exists r.N_{v_4},\ N_{X,X} \sqsubseteq \exists r.N_{v_1},\ N_{X,X} \sqsubseteq \exists r.N_{v_3}\},\\
\mathcal{T}_{\Pi_\Sigma} &\coloneqq \{N_{v_0} \equiv A \sqcap N_{A,X} \sqcap N_{A,A},\ N_{v_1} \equiv \exists r.N_{v_1 \cdot v_2},\ N_{v_1 \cdot v_2} \equiv N_{X,X},\ N_{v_3} \equiv B,\ N_{v_4} \equiv \top\}.
\end{aligned}$$

We now establish the correctness and completeness of our decision procedure.

Theorem 2. *Let $\mathcal{T}$ be a normalised $\mathcal{EL}$-TBox. Additionally, let Σ be a signature and let $A \in \Sigma$. Finally, let $\mathcal{G}_\Sigma = reduct_\Sigma(exp^{\mathcal{T}}_{\to}(A)) = (\mathcal{V}, \mathcal{E}, \mathcal{L}, \rho)$ and let $\mathcal{G}^{\Sigma,A}_{desc} = exp^{\mathcal{T}'}_{\to}(A)$ for $\mathcal{T}' \coloneqq norm(\mathcal{T}^{\Sigma,A}_{desc})$. Then the following two statements are equivalent*

(i) there exists a Σ-subsumer interpolant of A w.r.t. $\mathcal{T}$;

(ii) $sim_{\to}(\mathcal{G}_\Sigma, \mathcal{G}^{\Sigma,A}_{desc})$.

For the proof of the implication $(ii) \Rightarrow (i)$, we first observe that every $X \in \text{sig}^{\mathsf{N_C}}(\mathcal{T}^{\Sigma,A}_{\text{desc}}) \setminus \Sigma$ can be replaced with a concept $C \in \mathcal{EL}_\Sigma$ such that $\mathcal{T}' \models X \equiv C$. In that way we can construct a TBox $\mathcal{T}^{\Sigma}_{\text{s-ui}}$ such that $\text{sig}(\mathcal{T}^{\Sigma}_{\text{s-ui}}) \subseteq \Sigma$, and for every $\alpha = C \sqsubseteq D$ with $C, D \in \mathcal{EL}_\Sigma$ it holds that $\mathcal{T}^{\Sigma,A}_{\text{desc}} \models \alpha$ iff $\mathcal{T}^{\Sigma}_{\text{s-ui}} \models \alpha$. Then, as $\text{sim}_{\to}(\mathcal{G}_\Sigma, \mathcal{G}^{\Sigma,A}_{\text{desc}})$ holds by assumption, it follows that $\mathcal{T}^{\Sigma}_{\text{s-ui}}$ is a Σ-subsumer interpolant of A w.r.t. $\mathcal{T}$.

The implication $(i) \Rightarrow (ii)$ requires an involved proof, of which we can only give a high-level overview due to space constraints. As by assumption a Σ-subsumer interpolant $\mathcal{T}^{\Sigma}_{\text{s-ui}}$ of A w.r.t. $\mathcal{T}$ exists, let $\mathcal{G}^{\Sigma}_{\text{s-ui}} \coloneqq \text{reduct}_\Sigma(\text{exp}^{\mathcal{T}''}_{\to}(A))$ for $\mathcal{T}'' \coloneqq \text{norm}(\mathcal{T}^{\Sigma}_{\text{s-ui}})$. By definition of subsumer

	$\lvert\mathrm{sig}(\mathcal{T})\rvert$ $\mathsf{N_C}$/ $\mathsf{N_R}$	$\lvert\Sigma\rvert$ $\mathsf{N_C}$/ $\mathsf{N_R}$	Success Rate (%)	Subsumer UI exists? Yes / No	Average Time (s)
COGAT	1702/6	100/6	100	10/90	17.67
		150/6	100	64/36	8.57
CLO	5474/13	200/13	100	100/0	6.65
		300/13	100	100/0	5.94
ChEBI	102613/9	100/9	99	59/40	9.95
		200/9	100	71/29	16.29
NCI	118158/92	50/50	96	84/12	24.90
		100/50	96	82/14	73.09
		200/50	89	73/16	164.85
GALEN	23141/404	20/20	97	97/0	36.21
		30/30	95	93/2	54.52
		75/75	77	74/3	100.12

Table 1: Performance of our prototype implementation

interpolants, there exists a maximal (w.r.t. $\subseteq$) subsumer simulation S_1 between $\mathcal{G}_\Sigma$ and $\mathcal{G}^\Sigma_{\text{s-ui}}$, as well as a maximal subsumer simulation S_2 between $\mathcal{G}^\Sigma_{\text{s-ui}}$ and $\mathcal{G}_\Sigma$. One can then show that the composition $S := S_1 \circ S_2$ together with a special condition on the edges induces a subgraph $\mathcal{G}'_\Sigma$ of $\mathcal{G}_\Sigma$ (by restricting $\mathcal{G}_\Sigma$ to the nodes v such that $(v, v) \in S$) in which every cycle is Σ-prime and for which there exists a subsumer graph simulation S between $\mathcal{G}_\Sigma$ and $\mathcal{G}'_\Sigma$ with $(v', v') \in S$ for every node v' in $\mathcal{G}'_\Sigma$ (cf. graph $\mathcal{G}'_\Sigma$ in Figure 1 and Example 3). One can show that $\mathrm{sim}_\rightarrow(\mathcal{G}'_\Sigma, \mathcal{G}^{\Sigma,A}_{\text{desc}})$ and consequently, $\mathrm{sim}_\rightarrow(\mathcal{G}_\Sigma, \mathcal{G}^{\Sigma,A}_{\text{desc}})$ holds.

Once it is known that a Σ-subsumer interpolant exists, it can be constructed by unfolding the TBox $\mathcal{T}^{\Sigma,A}_{\text{desc}}$ as described in the proof sketch above for the implication $(ii) \Rightarrow (i)$. Moreover, it can easily be seen that $\mathcal{T}^{\Sigma,A}_{\text{desc}}$ completely describes $\mathcal{G}_\Sigma$ when $\mathcal{G}_\Sigma$ does not contain a cyclic path. We can thus infer that a subsumer interpolant always exists if $\mathcal{G}_\Sigma$ is acyclic.

As the existence of subsumer simulations can be decided in polynomial time in the size of the input graphs, we obtain the following complexity result.

Corollary 1. *Let $\mathcal{T}$ be a normalised $\mathcal{EL}$-TBox, let Σ be a signature and let $A \in \Sigma$. Then it can be decided in exponential time in the size of $\mathcal{T}$ and Σ whether there exists a Σ-subsumer interpolant of A w.r.t. $\mathcal{T}$.*

4 Experimental Evaluation

We conducted an initial experimental evaluation of a prototype implementation of our decision procedure on five real-world ontologies to assess the practical feasibility of our method. More precisely, given an $\mathcal{EL}$-TBox $\mathcal{T}_{\text{in}}$, a signature Σ, and a concept name $A \in \Sigma$ as input, in our prototype implementation the normalisation $\mathcal{T}$ of $\mathcal{T}_{\text{in}}$ and the Σ-reduct of the subsumer graph $\mathcal{G} := (\mathcal{V}, \mathcal{E}, \mathcal{L}, \rho)$ of A w.r.t. $\mathcal{T}$ is computed first. Subsequently, the Σ-prime interpolation TBox $\mathcal{T}^{\Sigma}_{\rightleftarrows}$ of $\mathcal{T}$ is constructed, which is followed by the computation of the candidate TBox $\mathcal{T}' = \mathcal{T}^{\Sigma,A}_{\text{desc}}$. Finally, the subsumer graph $\mathcal{G}^{\Sigma,A}_{\text{desc}}$ of A w.r.t. $\mathrm{norm}(\mathcal{T}^{\Sigma,A}_{\text{desc}})$ is constructed and our implementation checks whether $\mathrm{sim}_\rightarrow(\mathcal{G}, \mathcal{G}^{\Sigma,A}_{\text{desc}})$ holds.

As ontologies we selected the version 0.3 of the Cognitive Atlas Ontology (COGAT); release January 6, 2016 of the Chemical Entities of Biological Interest (ChEBI) ontology; version 2.0 of the Cell Line Ontology (CLO); the version of the Full-GALEN ontology without functional, inverse and transitive roles; and version 16.03d of the National Cancer Institute Thesaurus (NCI). We first removed every non-$\mathcal{EL}$ axiom from each ontology (including role inclusions).

In our experiments we focused on forgetting most of the signature symbols of the TBox, which corresponds to the use case of exhibiting hidden relationships between signature symbols using uniform interpolation, for instance. We selected the input concept names A and the input signatures Σ with up to 300 concept names and 75 role names that we used in our experiments as follows. For each considered signature size x/y (number of concept/role names) we selected concept names A from the respective ontologies $\mathcal{T}$ such that the subsumer graph $\exp_{\rightarrow}^{\mathcal{T}}(A)$ is cyclic and such that more than x concept names occur in $\exp_{\rightarrow}^{\mathcal{T}}(A)$. We only chose concept names A that yield cyclic subsumer graphs $\exp_{\rightarrow}^{\mathcal{T}}(A)$ as subsumer interpolants of acyclic subsumer graphs always exist. We then extracted $x - 1$ concept names and y role names of those that occur in $\exp_{\rightarrow}^{\mathcal{T}}(A)$ at random. For each signature size x/y we repeated the process described above 100 times.

All experiments were run on PCs equipped with an Intel Core i5-4590 CPU running at 3.30GHz. 16 GiB of heap space were allocated to the Java VM and an execution timeout of 10 CPU minutes was imposed on each problem.

The results of our experiments are shown in Table 1. For the fairly small ontologies COGAT and CLO, we did not encounter any timeouts and the existence of subsumer interpolants could typically be decided within a few seconds. We obtained similar results for the larger ontology ChEBI, which is, however, composed of axioms that have a simple structure to which Lemma 1 applies. Our tool has hence been able to decide the existence of uniform interpolants of ChEBI. Note that subsumer interpolants of CLO were found to exist in all of our test cases.

For the more complicated ontologies NCI and GALEN, we could observe that with increasing signature size the average computation time was also increasing and that the success rate was slightly decreasing. In most of the cases, our implementation could determine the existence of subsumer interpolants within a few minutes, even for GALEN, which is renowned for its complicated cyclic structure. For NCI the number of **no**-answers remained fairly stable throughout the various signature sizes, but for GALEN we only started to observe a small number of **no**-answers for signature sizes of 30 concept and role names.

The experiments show that deciding the existence of Σ-subsumer interpolants is feasible in practice despite our prototype not having an efficient implementation for determining the existence of graph simulations. In most of our test cases that did not result in a timeout the simulation check required more time than the construction of the TBox $\mathcal{T}_{\text{desc}}^{\Sigma,A}$.

5 Conclusion

We have shown how the problem of deciding the existence of a uniform interpolant of an $\mathcal{EL}$-TBox w.r.t. a signature can be divided into three subproblems. Here we have investigated the existence of subsumer interpolants of $\mathcal{EL}$-TBoxes for a concept name w.r.t. a signature, which is one of those subproblems. We have characterised the existence of subsumer interpolants using the proof-theoretic approach from [7]. We believe that the techniques presented in this paper can lead to a fundamentally new approach to deciding the existence of uniform interpolants of $\mathcal{EL}$-TBoxes. Moreover, we have presented an evaluation of a prototype implementation for deciding the existence of subsumer interpolants (and computing subsumer interpolants if they exist), and we demonstrated its viability in practice on several real-world ontologies, including the prominent ontologies NCI and GALEN. Our implementation can decide the existence of uniform interpolants in some cases as subsumer interpolants and uniform interpolants may coincide, which is the case for the ($\mathcal{EL}$-fragment of the) ontology ChEBI. To the best of our knowledge, our implementation is the first tool capable of deciding the existence of uniform interpolants, albeit limited to ontologies of a special form.

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