Evidence of Multi-Dimensional Herfindahl Hirschman Index from Linear Additive Directional Distance Function

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Abstract

Since its inception, application of Directional Distance Models has been found in abundance. Such concepts are invaluable for assessing performance of a firm in the midst of other rivals. Ample directions have been developed to satisfy a diverse set of criteria. The extant research is aimed to fulfil the sole objective of investigating and obtaining an inherent Direction Vector emerging from the Directional Distance Additive Model (DDAM). In this process, the existence of a Multi-Dimensional Herfindahl Hirschman Index (MHHI) is evidenced. The first Eigen-vector of MHHI is proved to be legitimate owing to its fulfilment of properties to symbolize a Directional Distance vector. This newly devised vector possesses the merit of corroborating the competitive position of a set of firms. In this regard, the output oriented form of DDAM is designed to foretell the volume of desirable outputs to be escalated in view of attaining a superior position in the selling market while implicating the same amount of resources. Principal Component Analysis plays a key role to identify the output-oriented directions from the non-central covariance matrix (MHHI) obtained from the output vectors.

Keyword:

Multi-Dimensional Herfindahl Hirschman Index, Directional Distance Additive Model, non-central covariance matrix, Principal Component Analysis

1. Introduction:

Like Data Envelopment Analysis (DEA), Distance functions are meant for measuring efficiency. According to Russell (1998), the first formulation of the distance function was due to Debreu (1951) who referred to it as "coefficient of resource utilization". However, duality concepts were mentioned later on by Aparicio P. and Pastor J. T. (2010). It was Shephard (1953) (Ray S. C. (2004)) who introduced the concept of multiplicative type Distance Functions along with the duality principles. Chambers, R.G., Chung, Y., Fare, R., (1996) explored a relationship among the distance function created by Shephard (1953) and the Benefits function of Luenberger D. G. (1992). This function was different from the earlier multiplier based models of Shephard (1953) or a gauge function originated by McFadden (1978). The duality principles were also elaborated. Later on, Chambers, R.G., Chung, Y., Fare, R., (1998), added a few theoretical concepts to evaluate the Nerlovian Efficiency and Technical Efficiency scores from the same additive model. However, such studies did not refer to any matrix to spell out market concentration or competition.

Aparicio P. And Pastor J. T. (2010) introduced of two types of DDF models such as a multiplicative type ratio directional distance function (Malmquist Index based) and an additive type based on linear distance functions. In another theoretical development, Aparicio P. and Pastor J. T. (2012) formulated a DDF based on the assumptions of the rate of return subjected to two more conditions like known or unknown input factor prices. Farrell's performance score was adopted in the first scenario. Three expressions were provided to understand the technical inefficiency, regulatory inefficiency, and allocative inefficiency. DDF was used for the later one for making a choice of reference and the dual form of it was assumed as an alternative input price. Cheng G., ZervoPoulos P. D. (2012) proposed a unit invariant generalized DDF method for handling both desirable and undesirable outputs to evaluate the performance of 160 national health systems. The authors incorporated a special constraint in the DDF model to demarcate the bad outputs from the good ones. They also suggested a generalized form of efficiency score which was akin to the slack-based models.

Considering the priority of earning more profit for a firm, Zofio, J. L. et al (2012) proposed a profit inefficiency based direction vector which could detect the technological or allocative problems within a firm. Fare, R. et al (2013) showed the way to use the slack-based models to estimate the optimal output based direction functions for developing endogenous direction vectors. Atkinson, S. E. & Tsionas, M. G., (2016) adopted a quadratic distance function depending on the good as well as bad inputs and outputs. The unknown parameters were determined from the properties described earlier by Hudgins & Paramount (2007). Daraio, C. & Simar, L. (2016), on the contrary, adopted a data-driven approach to gauge the inefficiency of a firm among a group of heterogeneous firms. Authors argued that the egalitarian direction, which was proposed by Fare, R. et al (2008), may lead to more weight on the big input inefficient firms. Averaging on the basis of angles was found valuable for measuring efficiency scores.

Authors (Wang Ke., et al., (2017)) reviewed and classified the DDF approaches into two segments termed as the Exogenous (arbitrary vectors) and Endogenous (Theoretically Optimized directions such as closest location of the peer or driven by market-oriented potentials such as maximization of profit etc) DDF techniques. Fare R. et al (2017) estimated the parameters within a non-linear model to derive endogenous directions which had a remarkable impact in selecting a peer at least possible distance in comparison to the directions obtained from using econometric methods and Data Envelopment Analysis.

In the context handling negative data, the directional distance model, prescribed by Chambers, R. G. et al., (1998), was applied by Portela et al. (2004) for measuring performances of all branches of a Portuguese bank. The efficiency of a firm is measured in comparison to the deviation seen from a pre-defined Ideal point (Superior Origin). In this regard, another exemplary radial oriented unit invariant model proposed by Cheng, G. et al (2011) can be included under the category of Directional Distance Function which was able to achieve similar outputs as obtained from other radial models.

Unlike previous applications, the extant model presents a major way-out to remain more competitive in a market by reorienting the *DDF* proposed by Chamber et al (1998). **Competition** is the rivalry between companies selling similar products and services with the goal of achieving revenue, profit, and market share growth. The nature of competition in the market has been implicated with the Number of rivals, Similarity among the products, Ease of entry or exit, Unionization and Control over prices. In this paper the competition is presumed to be prevailing among a set of rivals during the selling of a number of substitutes in a single or a group of markets while consuming similar types of inputs within a particular span of time. The model is manoeuvred to develop a unique output oriented direction which is able to strengthen the market position of a firm (to have a better grip in the market) by optimizing its outputs in a meticulous manner for certain level of input utilization.

The term *Market Concentration* claims a vital spot in this regard. High concentration implies that, in a market, a large proportion of the economic resources and activities are controlled by a small number of firms. This eventually makes them powerful to put control of the price. As a result, the extent of competition in the market gets attenuated. Entry of a firm into the market or its exit from it, or a merger can induce changes in concentration to influence the extent of competition and performance of the firms in the market.

Concentration Indexes give an appraisal of concentration degree and inequality of market share among companies that operate in a market. They are able to provide a relatively clear image of companies' potential to use market power in pricing, in determining volume and quality of products and services. Researchers, according to the weighting scheme, recommended ten concentration ratios such as – the *k* bank Concentration Ratio (*CRk*); the Herfindahl-Hirschman Index (*HHI*); the Hall-Tideman Index (*HTI*); the Rosenbluth Index (*RI*); the Comprehensive Industrial Concentration Index (*CCI*); the Hannah and Kay Index (*HKI*); the U Index (*U*); the multiplicative Hause Index (*Hm*); the additive Hause Index (*Ha*); and the Entropy measure (*E*). The most frequently used indices are *CRk* and *HHI*.

Albert Hirschman recognized the limitations of availing complete data or computing true market power or using the Lorenz-Gini methodology in an industrial organization context. In his book (1945) on international trade, he pointed out that a measure of concentration should be composed of equality of market shares and the number of total competitors. He conceived of an index, referred as HHI, to evaluate market concentration which had to increase owing to the increase in the dispersion of the market share and had to decrease owing to the presence of a large number of firms.

$$HHI = \frac{100}{Q^2}\sum_{i=1}^{c}(q_i)^2 \ \ where\ Q = \sum_{i=1}^{c} q_i$$

'Market Share' is normally used as q_i but sales, employment statistics, number of people using a company's services, number of outlets, etc can be used as surrogates. The inverse of this index is referred to as the equivalent number of equal-sized firms which are also called as effective competitors in the industry. The index may rise as high as 10,000 if the market has a monopoly to imply no competition. On the contrary, the scenario of perfect competition is reflected by an index value zero. There has been wide range of applications in diverse fields such as Lijesen *et. al* (2002), Saribas & Tekiner (2015), Susilo & Axhaunsen (2014), etc undertook HHI to measure the extent of competition in the aviation sector. Akdoğu & MacKay (2008) emphasized that investment depends on industry concentration. However, Lee & Hwang (2003) did not come across any associations between market structure determinants and investment decisions in the Korean telecommunication industry. Mateo, C. L., Porras, A. R., (2010) were able to establish an association between Investment decisions and Market Concentration. White, L. J. (1987) mentioned about two levels of HHI to speculate the likelihood of getting a challenge against a merger or not. According to the author, "if entry conditions are easy, virtually any merger would be allowed; with high entry barriers, mergers in the middle and upper HHI bands receive closer scrutiny". He proposed a level of 1600 instead of 1800 for fixing the upper decision point to offer a healthier defence. He clearly stated the increased size of the business during the post-merger session can bring an improved level of efficiency. Furthermore, it has also been a legislative measure to evaluate the impact of a proposed merger in the US banking industry.

Kar & Swain (2014) used the structure-conduct-performance (SCP) approach (e.g. Herfindahl-Hirschman index (HHI)) to measure competition in the microfinance industry of 71 countries between 2003 and 2008. Assefa et al. (2013) had a similar finding which stated greater competition in the South Asian microfinance industry. However, in spite of such wide range of application HHI has few drawbacks. For example,

- There is an inherent inability of indicators to acknowledge certain qualitative characteristics of the market, such as market structure stability, the level of product differentiation, the height of entry barriers, operating cost, etc.
- this index does not include industrial tradition, as well as features and objectives of managers who run the market that is being analyzed.
- the same value of the indicator will not have the same meaning for "small" and "large" economy. In a small economy owing to the small space and low purchasing power there can be a higher level of tolerance to the high value of concentration indicator. In such a scenario, a small number of companies operate in a limited market.
- a scalar form of HHI fails to analyze the concentration for a set of products performing in a large number of markets
- products with cross elasticity cannot be treated well with the aid of HHI

The whole research is divided into two phases. At its first phase, an input output data, closely associated with the *suppliers and customers of a set of rival firms*, is to be gathered to embed the notion of market competition among them. The number of competitors and their volume of resource consumption to produce sellable outputs are effectively used to assess the extent of competition through Competitive Intensity. In this regard, the basic framework of the *DDF* proposed by Chamber et al (1998) is revisited to obtain a unique matrix termed as Multi-Dimensional Herfindahl-Hirschman Index (MHHI). This matrix is found to be essential to explicate the extent of competition (which is akin to HHI) in a market even in its multidimensional form and can eradicate many shortcomings of the scalar form of

Competitive Index known as Herfindahl-Hirschman Index. Lerner's Theory is revisited for the sake of framing a relationship between the index to measure market power and the presence of several product categories in a single or multiple markets. As the whole matrix of MHHI contains HHI values of the individual products on its diagonal and keeps the association terms on non-diagonals due to the substitution effect among them. Various Eigen-values and Eigenvectors (Endogenous directions) obtained from MHHI play vital role to analyse the characteristics of various known market structures. In the second phase, it answers the major inquisition about the viability of these derived directions along which the progress is to be made. This movement will project an inefficient firm onto the production frontier to become technologically efficient and also to ensure a better competitive position in the market. The whole episode is aimed to apply the model of Chamber et al (1998) in an innovative manner and hence does not possess any background survey.

2. Prevalent Directional Distance Additive Model:

The basic aim of a Directional Distance Additive Model is to extract efficient performers from a set of Decision Making Units. The extent of inefficiency is measured on a fixed or variable Direction fulfilling a set of purposes. In this context, a technology set (Production Possible Set) *T is conceived,* which consumes an input vector x to produce an output vector y (as described below):

$$T = \{(x, y): x\ can\ produce\ y\}\ \ where\ x \in R^v, y \in R^m$$

Few essential assumptions are adopted for ensuring the members within the set *T.*

- If there is an activity (x_0, y_0) such that, $(x_0, y_0) \in T,$ then another activity (x_0, y) will also be an element of *T* due to free disposability of outputs $(y \leq y_0)$.
- If there is an activity (x_0, y_0) such that,$(x_0, y_0) \in T,$ then any activity (x, y_0) will also be an element of *T* due to free disposability of inputs $(x_0 \leq x)$.
- *T* is a convex set
- If *G* is an output set of efficient DMUs, then, $T \supset G$ and for a parameter θ^*, from the first formulation the activity $(x - \theta^* g_x, y)$ must be within *G*. θ^* is needed to be obtained from the subsequent relationship

$$\theta^* = max\{\theta: so\ that\ (x - \beta g_x, y) \in G\ \&\ (x, y) \in T, x = [x_1 \quad x_2 \quad \cdots \quad x_v]^T\}$$

Apart from the three basic postulates stated before an additional property of the output-oriented form of the technology set *T,* is mentioned below:

- The input set G' of efficient DMUs, for a parameter μ^*, must contain the activities mentioned by the combination $\left(x, y+\mu^* g_y\right)$. The optimum value of μ^* is derived from the following optimization problem:

$$\mu^* = max\left\{\mu: for\ \left(x, y+\mu g_y\right) \in G' \& (x, y) \in T, y = [y_1 \quad y_2 \quad \cdots \quad y_m]^T\right\}$$

$\theta^*\ and\ \mu^*$ described here behave like the directional distance functions proposed by Chambers et al., (1996), Chambers et al., (1998) as depicted below:

$$D\left(x, y, g_x, g_y\right) = max\left\{\beta: \left(x-\beta g_x, y+\beta g_y\right) \epsilon T\right\} \quad (1)$$

β, here, is treated as the extent of inefficiency measured along the vector $\left(g_x, g_y\right)$.

In these circumstances, the model, however, does not attempt to associate any suppliers' or buyers' market to acknowledge the sourcing or distributing quantity of goods. Hence, it can be presumed that the aggregated volume of inputs or outputs were coupled with a single market. Then, quite evidently, a firm will look forward to come up with strategies to become a market leader. Hence, the set of activities will be a mere mirror image of strategies opted.

2.1. Output Oriented Linear Additive DDF and its Inherent Direction Vector:

The general directional distance model under Variable Return to Scale (VRS) is expressed as

$$\text{Max}\, \beta$$

$$x\lambda + \beta g_X \leq x_0$$

$$y\lambda - \beta g_Y \geq y_0$$

$$\lambda = [\lambda_1 \quad \lambda_2 \cdots \quad \lambda_c]^T$$

$$\sum_{i=1}^{c} \lambda_i = 1, for\ i = 1,2 \ldots c \quad (2)$$

The functions mentioned above as $g_X\ and\ g_Y$ are basically the indicators of the direction of improvement. This non-oriented model can be transformed into input (output) oriented model by allowing $g_Y = 0$ ($g_x = 0$). Though, a Directional distance function model can deal with negative data in under VRS (due to translation invariance), but, it is neither capable of showing unit invariance property nor it scores within in between 0 and 1.

The output-oriented model is obtained by setting $g_X = 0$ in equation (2). The performance measure (β_0) of a DMU-O is computed by solving the subsequent optimization problem

$$\max \beta_0$$

$x\lambda \leq x_0$

$y\lambda - \beta_0 g_y \geq y_0, \; s.t. \;\; g_y > 0,$

$\lambda = [\lambda_1 \quad \lambda_2 \cdots \quad \lambda_c]^T$

$\sum_{i=1}^{c} \lambda_i = 1$ (3)

The second constraint of model (3) is rearranged for the sake of obtaining an endogenous direction which is referred as *an output oriented inherent direction*.

$y\lambda - \beta_0 g_y \geq y_0$

$y\big(\lambda - \beta_0 y^T (yy^T)^{-1} g_y\big) \geq y_0$ (4)

Any direction g_y is conceived as a feasible one if a set of properties is being pursued (illustrated in Appendix 2). According to these properties, both β_0 and g_y should essentially stay nonnegative. Constraint (4) is essentially a stepping stone for issuing a feasible inherent direction for this problem. Let it be assumed that there exists g_y and γ_y ($\gamma_y > 0$ & $g_y > 0$) such that they remain to be the Principal Eigenvector and Eigen value of $(yy^T)^{-1}$ associated with g_y. As a consequence equation (5) will be satisfied:

$(yy^T)^{-1} g_y = \gamma_y g_y$ (5)

Combination of (4) and (5) gives rise to the same constraint mentioned in (2) (shown below):

$y\left(\lambda - \gamma_y \beta_0 y^T g_y\right) \geq y_0$

$y\lambda - \gamma_y \beta_0 (yy^T) g_y \geq y_0$ (6)

But, (5) approves the equivalence denoted as $(yy^T) g_y = \frac{g_y}{\gamma_y}$. Hence, (6) will be reduced to its original form mentioned in (4).

$y\lambda - \gamma_y \beta_0 \frac{g_y}{\gamma_y} \geq y_0$

$y\lambda - \beta_0 g_y \geq y_0$ (7)

In a nutshell, any nonnegative Eigenvector computed from the matrix $(yy^T)^{-1}$ can be a feasible direction for the output oriented Linear Directional Distance Model. However, according to Sarkar, S. (2014), the first Principal Eigen vector of (yy^T) certainly has the nonnegative property. Multiplying both sides of (5) with (yy^T) establishes the relationship of $(yy^T)g_y = \frac{g_y}{\gamma_y}$. As a result, the first Eigenvector of (yy^T) emerges as a feasible solution of the output oriented problem. Moreover, (yy^T) is a positive definite symmetric matrix. This recommended unit vector is indeed responsible for explaining largest possible non-centred variation of outputs and therefore can certainly have important features to describe the competitive position of a set of firms.

2.2. Input Oriented Linear Additive DDF and its Inherent Direction Vector:

An input oriented model can be obtained by setting $g_Y = 0$ in the equation (2). Pursuing the steps referred above, the magnitude of g_x is determined from the nonnegative Eigen Vector (E_{1x}) of the matrix (xx^T). The performance measure (β_0) of a DMU-O is computed by solving the subsequent optimization problem:

$\max \beta_0$

$x\lambda + \beta_0 E_{1x} \leq x_0,\ s.t.\ \ E_{1x} > 0$

$y\lambda \geq y_0$

$\lambda = [\lambda_1 \quad \lambda_2 \cdots \quad \lambda_c]^T$

$\sum_{i=1}^{c} \lambda_i = 1$ (8)

Eigenvectors computed above do possess similar properties as pursued by the directions prescribed in any Linear DDF model (Appendix 1). In the circumstances of meeting the demand forecast, this model plays a key role for prescribing the way of economic resource utilisation.

2.3. Absence of Inherent Direction for a Non-Oriented Model:

Reiterating the model mentioned in (2) the non-oriented form of the DDF is displayed below:

$\max \beta$

$x\lambda + \beta g_X \leq x_0$

$y\lambda - \beta g_Y \geq y_0,\ g_X, g_Y > 0$

$\lambda = [\lambda_1 \quad \lambda_2 \cdots \quad \lambda_c]^T$

$\sum_{i=1}^{c} \lambda_i = 1$ (9)

Both input and output constraints are manoeuvred to recognize the Inherent direction.

$$\begin{bmatrix} x \\ -y \end{bmatrix} \lambda + \beta \begin{bmatrix} g_X \\ g_Y \end{bmatrix} \leq \begin{bmatrix} x_0 \\ -y_0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ -y \end{bmatrix} \left(\lambda + \beta [x^T \quad -y^T] \left(\begin{bmatrix} x \\ -y \end{bmatrix} [x^T \quad -y^T] \right)^{-1} \begin{bmatrix} g_X \\ g_Y \end{bmatrix} \right) \leq \begin{bmatrix} x_0 \\ -y_0 \end{bmatrix}$$

$$\left(\begin{bmatrix} x \\ -y \end{bmatrix} [x^T \quad -y^T] \right)^{-1} \begin{bmatrix} g_X \\ g_Y \end{bmatrix} = \begin{bmatrix} xx^T & -xy^T \\ -yx^T & yy^T \end{bmatrix}^{-1} \begin{bmatrix} g_X \\ g_Y \end{bmatrix} \leq \begin{bmatrix} x_0 \\ -y_0 \end{bmatrix} \qquad (10)$$

Unlike xx^T or even yy^T the matrix to be handled in (10) is not at all a positive definite. Thus, it is the first Eigen Vector does not contain positive elements throughout to violate the nonnegative property of the Eigenvector $[g_X \quad g_Y]^T$. Though, there may be few instances for which a nonnegative Eigen Vector may appear having a meagre amount of explaining power of the total variation. These vectors should be relinquished as they lack competitive intensity.

3. Inherent Direction Indicating Competitive Intensity:

Traditionally, a firm aspires to maximise its output set from within the available input set for gaining more market share. Hence, the output oriented model is prescribed for those circumstances where the production of more volume can be sold in the market. In other words, this solution will certainly suggest the inefficient firms to improve market share of a set of outputs. Contrarily, the input-oriented model is preferred during the slump of demand. Under this scenario, a legitimate question may be raised that can there be an output oriented solution which offers the inefficient firms a way to improve its market share to earn a superior competitive position. In short, how can the supremacy of the efficient firms be neutralised by reducing the Herfindahl-Hirschman Index (*HHI*) for each product? *HHI* has been reckoned as one of the widely accepted tool to measure market concentration. This index has a strong implication with price-cost margin to explain market performance. The value of the index is computed while considering the number of firms along with the size of the individual firms. For example, if 3 (30%), 2 (20%) & 5 (50%) are the units (percentage) of a product produced & marketed by A, B and C respectively then as per the definition *HHI* will be same as follows:

Non-normalised $HHI = 3^2 + 2^2 + 5^2 = 38$, Normalised $HHI = 30^2 + 20^2 + 50^2 = 3800$

Hence, a multi-dimensional *HHI* (HHI_o) is required to be proposed to include all products within mentioned in the output oriented model. The subsequent sections will elaborate the composure of yy^T to be referred as a multi-dimensional *HHI* (HHI_o).

3.1. Generalized expression of a multi-dimensional Non-normalised HHI:

Let there be a market of *c* number of firms which are connected with *s* number of sellers and *b* number of buyers. Any i^{th} firm possesses an input-output vector of $(X_i,\ Y_i)$.

The input-output vector is further expanded to include all buyers and sellers.

$$X_i{}^T = \begin{bmatrix} x_{11i} & x_{12i} & \cdots & x_{1si} \\ x_{21i} & x_{22i} & \cdots & x_{2si} \\ \vdots & \vdots & \ddots & \\ x_{p1i} & x_{p2i} & & x_{psi} \end{bmatrix} \text{ and } Y_i{}^T = \begin{bmatrix} y_{11i} & y_{12i} & \cdots & y_{1bi} \\ y_{21i} & y_{22i} & \cdots & y_{2bi} \\ \vdots & \vdots & \ddots & \\ y_{o1i} & y_{o2i} & & y_{obi} \end{bmatrix}$$

In these expressions, x_{11i} (y_{11i}) implies the volume of 1st input supplied (1st output delivered) by the 1st supplier delivered (to the 1st buyer) to the i^{th} firm (from the i^{th} firm). Multi-dimensional input and output oriented HHI are categorically described from $X_i{}^T$ & $Y_i{}^T$:

$MHHI_I = xx^T = \sum_{i=1}^{c} X_i{}^T X_i$ and $MHHI_o = yy^T = \sum_{i=1}^{c} Y_i{}^T Y_i$

3.2. Generalized expression of a multi-dimensional Normalised HHI

Continuing with the same market of *c* number of firms which are connected with *s* number of sellers and *b* number of buyers the Normalised MHHI is explained below. Any i^{th} firm having an input-output vector of $(X_i,\ Y_i)$ is further expanded to include all buyers and sellers.

$$PX_i{}^T = \begin{bmatrix} \frac{x_{i11}}{X_{11}} & \frac{x_{i12}}{X_{12}} & \cdots & \frac{x_{i1s}}{X_{1s}} \\ \frac{x_{i21}}{X_{21}} & \frac{x_{i22}}{X_{22}} & \cdots & \frac{x_{i2s}}{X_{2s}} \\ \vdots & \vdots & \ddots & \\ \frac{x_{ip1}}{X_{p1}} & \frac{x_{ip2}}{X_{p2}} & & \frac{x_{ips}}{X_{ps}} \end{bmatrix} \text{ and } PY_i{}^T = \begin{bmatrix} \frac{y_{i11}}{Y_{11}} & \frac{y_{i12}}{Y_{12}} & \cdots & \frac{y_{i1b}}{Y_{1b}} \\ \frac{y_{i21}}{Y_{21}} & \frac{y_{i22}}{Y_{22}} & \cdots & \frac{y_{i2b}}{Y_{2b}} \\ \vdots & \vdots & \ddots & \\ \frac{y_{io1}}{Y_{o1}} & \frac{y_{io2}}{Y_{o2}} & & \frac{y_{iob}}{Y_{ob}} \end{bmatrix}$$

Apart from the usual terms like x_{i11} & y_{i11}, the total volume of 1st input (1st output) sold by the 1st supplier (firm itself) is denoted as X_{11} (Y_{11}). In view of more flexibility in the definition, the buyers and sellers can be replaced by markets from which the inputs are purchased and to which the goods are sold. Hence, a normalised multi-dimensional input and output oriented HHI is denoted as follows:

$MHHI_I = \sum_{i=1}^{c} PX_i^{\,T} PX_i$ and $MHHI_O = \sum_{i=1}^{c} PY_i^{\,T} PY_i$

The formation of Normalised and Non-normalised MHHI is elaborated in the subsequent example. Let there be three firms A, B and C engaged in producing & selling two outputs in different volumes in the same market. If (3, 6), (2, 5), (4, 5) are the combinations produced & marketed by A, B and C respectively then as per the definitions the mathematical representations are mentioned below:

$Y_A = \begin{bmatrix} 3 \\ 6 \end{bmatrix}^T, Y_B = \begin{bmatrix} 2 \\ 5 \end{bmatrix}^T$ & $Y_C = \begin{bmatrix} 5 \\ 4 \end{bmatrix}^T$

Thus, Non-normalised $MHHI_o$ can also be visualised using these values.

$$MHHI_o = Y_A^{\,T} Y_A + Y_B^{\,T} Y_B + Y_C^{\,T} Y_C = \begin{bmatrix} 38 & 48 \\ 48 & 77 \end{bmatrix}$$

The element located at 1st row and 1st column is the Non-normalised HHI of the first product computed from squaring all the individual sales volumes ($3^2 + 2^2 + 5^2 = 38$) of A, B and C. Similarly, the Non-normalised HHI of the second product is found as 77 in the same matrix.

The Normalised version can be computed from the market share (in percentage) of each of these products measured in terms of $PY_A^{\,T}, PY_B^{\,T}$ & $PY_C^{\,T}$.

$PY_A^{\,T} = [30 \quad 40], PY_B^{\,T} = [20 \quad 33.3]$ & $PY_C^{\,T} = [50 \quad 26.6]$

$$MHHI_o = PY_A^{\,T} PY_A + PY_B^{\,T} PY_B + PY_C^{\,T} PY_C = \begin{bmatrix} 3800 & 3200 \\ 3200 & 3422.2 \end{bmatrix}$$

Normalised values of HHI due to the first & second products are 3800 & 3422 to specify a highly concentrated market for both of these products. The implication of the non-diagonal terms in the MHHI matrix is mentioned in Appendix 2. Existence of these elements is feasible due to the presence of cross elasticity between the products. So, similar category of substitutable products should be conceived under these circumstances.

3.3. Principal Component Analysis of the Eigen Vectors from *MHHI*:

Various types of market structures are elaborated in this segment in association with MHHI. The simplest assumption of an industry dominated by a single firm and selling only one product would comprise a scalar value one. The mandate from the HHI score would refer to it as the highest market concentration. If a producer serves a monopoly market with m number of products the corresponding MHHI will be described as follows:

$$S_1 = \begin{bmatrix} \frac{{y_{11}}^2}{{Y_1}^2} & \frac{y_{11}y_{21}}{Y_2Y_1} \cdots & \frac{y_{11}y_{m1}}{Y_mY_1} \\ \frac{y_{11}y_{21}}{Y_2Y_1} & \frac{{y_{21}}^2}{{Y_2}^2} \cdots & \frac{y_{11}y_{m2}}{Y_2Y_m} \\ \vdots & \ddots & \\ \frac{y_{11}y_{m1}}{Y_mY_1} & & \frac{{y_{m1}}^2}{{Y_m}^2} \end{bmatrix} = MHHI_O$$

Since there exists no other rival in this case so it leads to the equality $y_{i1} = Y_i \; for \; i = 1,2 \ldots c$. The resulting matrix S_1, therefore, retains only one within its each and every place. The subscript 1 is used to denote the monopolist.

$$S_1 = \begin{bmatrix} 1 & 1 \ldots & 1 \\ 1 & 1 \ldots & 1 \\ \vdots & \ddots & \\ 1 & & 1 \end{bmatrix}$$

On the contrary, if c numbers of equal-sized firms are involved in the same type of business then the total effect will be the derived from the sum $S_C = \sum_{i=1}^{c} S_i$ (Shown below):

$$S_C = \begin{bmatrix} \sum_{i=1}^{c} \frac{{y_{1i}}^2}{{Y_1}^2} & \sum_{i=1}^{c} \frac{y_{1i}y_{2i}}{Y_2Y_1} \cdots & \sum_{i=1}^{c} \frac{y_{1i}y_{mi}}{Y_mY_1} \\ \sum_{i=1}^{c} \frac{y_{1i}y_{2i}}{Y_2Y_1} & \sum_{i=1}^{c} \frac{{y_{2i}}^2}{{Y_2}^2} \cdots & \sum_{i=1}^{c} \frac{y_{mi}y_{2i}}{Y_mY_2} \\ \vdots & \ddots & \\ \sum_{i=1}^{c} \frac{y_{1i}y_{mi}}{Y_mY_1} & & \sum_{i=1}^{c} \frac{{y_{mi}}^2}{{Y_m}^2} \end{bmatrix} \text{ where } y_{1i} = \frac{Y_1}{c} \; for \; i = 1,2 \ldots c$$

$$S_C = \begin{bmatrix} \frac{1}{c} & \frac{1}{c} \cdots & \frac{1}{c} \\ \frac{1}{c} & \frac{1}{c} \cdots & \frac{1}{c} \\ \vdots & \ddots & \\ \frac{1}{c} & \frac{1}{c} \cdots & \frac{1}{c} \end{bmatrix}$$

On the other hand, an oligopoly structure can be studied through this *MHHI*. These matrixes, in the presence of a small number of strong players and dealing with undifferentiated products, are able to create a distinct scenario in comparison to the other two.

$$S_C = \begin{bmatrix} \sum_{i=1}^{c} \frac{{y_{1i}}^2}{{Y_1}^2} & \sum_{i=1}^{c} \frac{y_{1i}y_{2i}}{Y_2Y_1} \cdots & \sum_{i=1}^{c} \frac{y_{1i}y_{mi}}{Y_mY_1} \\ \sum_{i=1}^{c} \frac{y_{1i}y_{2i}}{Y_2Y_1} & \sum_{i=1}^{c} \frac{{y_{2i}}^2}{{Y_2}^2} \cdots & \sum_{i=1}^{c} \frac{y_{mi}y_{2i}}{Y_mY_2} \\ \vdots & \ddots & \\ \sum_{i=1}^{c} \frac{y_{1i}y_{mi}}{Y_mY_1} & & \sum_{i=1}^{c} \frac{{y_{mi}}^2}{{Y_m}^2} \end{bmatrix}$$

Single non-zero Eigen vector will attain a value of m if a monopoly competition is analysed. The remaining Eigen values will be zero and hence give rise to a single Eigen Vector.

In the context of perfect competition, the one and only non-zero Eigenvalue, in this case, will be having a value of $\frac{m}{c}$. This ratio approaches towards zero as c assumes a large value.

The matrix S_C owing to an oligopoly market with a small number of players characterizes according to two conditions such as $c > m \, or \, c < m$. The first condition will result m number of distinct Eigenvalues with same number of orthogonal Eigenvectors. But, the later one may produce at most c number of non-zero Eigenvalues.

3.4. Substitution of Products & *MHHI*:

Substitution among products deals with the effect of change in the price of one good on the demand of the other. Depending upon the nature of a decrease or increase in selling two types of goods such as substitute goods and complement goods appear from this analysis. Substitutes, under the changed condition, replace each other. On the other hand, compliments tend to move in the same manner owing to the changes made in the price of anyone. In this context, if it is assumed that the rate of percentage change found in sales volume (y_1) of product 1 due to the percentage change of price (p_2) of product 2 is CED then $CED = \frac{p_2}{y_1}\frac{dy_1}{dp_2}$.

The following example is incorporated to explain the impact of this effect. Let there be two rivals in a market competing with two unrelated products. The sales volumes of these competitors before and after the price change in product 1 are referred below:

$$Y_{1O} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}, Y_{1N} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, Y_{2O} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}, Y_{2N} = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$$

Y_{1O}, Y_{1N} are the two vectors to represent sold units of two products by the first seller before & after introducing the change (referred to as Old (O) and New (N)). The increase in the price of product 1 by the first seller observes a slump of demand from 4 units to 2 whereas the seller 2 finds an increase in demand in product 1 from 6 units to 8 units.

$$PY_{1_O}{}^T = \begin{bmatrix} \frac{y_{11}}{Y_1} \\ \frac{y_{21}}{Y_2} \end{bmatrix}_O = \begin{bmatrix} \frac{4}{10} \\ \frac{5}{7} \end{bmatrix} \text{ and } PY_{2_O}{}^T = \begin{bmatrix} \frac{y_{12}}{Y_1} \\ \frac{y_{22}}{Y_2} \end{bmatrix}_O = \begin{bmatrix} \frac{6}{10} \\ \frac{2}{7} \end{bmatrix}$$

$$PY_{1N}{}^T = \begin{bmatrix} \frac{y_{11}}{Y_1} \\ \frac{y_{21}}{Y_2} \end{bmatrix}_N = \begin{bmatrix} \frac{2}{10} \\ \frac{5}{7} \end{bmatrix} \text{ and } PY_{2N}{}^T = \begin{bmatrix} \frac{y_{12}}{Y_1} \\ \frac{y_{22}}{Y_2} \end{bmatrix}_N = \begin{bmatrix} \frac{8}{10} \\ \frac{2}{7} \end{bmatrix}$$

Hence, HHI obtained in these scenarios are displayed below:

$$HHI_O = \begin{bmatrix} 0.52 & 0.457 \\ 0.457 & 0.591 \end{bmatrix} \text{ and } HHI_N = \begin{bmatrix} 0.68 & 0.371 \\ 0.371 & 0.591 \end{bmatrix}$$

Changes can be observed in the Eigen Vectors computed from the Principal Component Analysis. Magnitudes of the first Eigenvalues in these two cases are respectively (1.015) &

(1.01). These changes will make an impact on the orientation of the First Eigen Vector (and also make a shift in Market Concentration).

3.5. Interpretation of Eigenvalues & Eigenvectors:

The precise connotation of the term Eigenvalue of a vector transformation is the measure of inflation or deflation of the same vector in the same direction or in the opposite direction. The constituent Vector is termed as an Eigenvector. As per the previous illustrations, variation in the Eigenvalue can be noticed due to Perfect competition and monopoly competition (in other words due to market concentration). With a variance of m the later one is ahead of the former one in comparison to $\frac{m}{c}$ (which even tends to 0 at a large value c). Hence, the variance keeps on increasing as market approaches towards a monopoly. The first Eigenvector of such a input based non-centred covariance matrix possesses a certain set of properties (mentioned in the sections of Sarkar, S., (2014)). Cost can be reduced if any improvement in the radial distance is achieved by pursuing through these directions. Contrarily, a vector derived from the output vector will lead to higher volume of production through a fixed product mix. In a nutshell, the entire scenario is identical to the outcomes that HHI used to produce for a single market.

3.6. Graphical Explanation Eigen Vectors:

Quoting from the earlier research of Sarkar, S. (2014 (a)), (2014 (b)), (2017), (2019)) Figure 1 is revisited to enumerate the importance of such Eigenvectors. The author has explained the concept of cost leadership and differentiation with the aid of these vectors without having any formal information about real time cost vector. The first Principal Eigen Vector (T1T2 in the figure) obtained from an input based un-centred covariance matrix is a determinant of the resource parsimony of a firm. Firm B was referred as a cost leader due to its minimum projection on the vector T1T2. The remaining Eigenvectors (here S1S2) are classified as the representatives of Differentiation. Followers of Differentiation strategy (such as A, C & D in Figure 1) possess capabilities to mix these resources to imbibe uniqueness in the products.

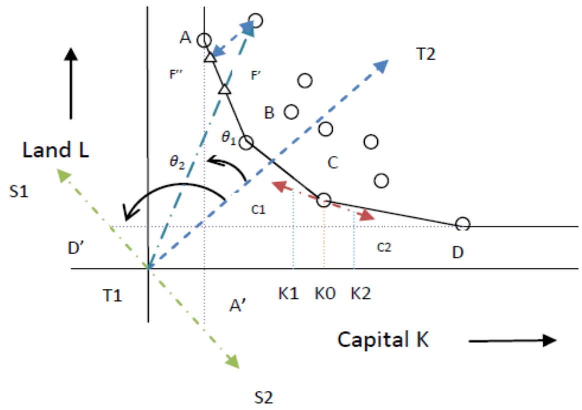

Figure 3.1: Production Frontier with Cost and Differentiation Ploy

On the contrary, the output based Eigenvectors will rationalise the revenue to be earned by the firms and the effective mix of outputs to be generated to gain superiority in the market.

3.7. Additional Benefits of Multi-Dimensional HHI:

The model seeks a set of representative variables instead of a single variable "market share". The analysis of a multi-dimensional HHI could reveal

- various alternative directions to display the market power and hence the degree of similarities among the competing firms due to them.
- the impact of five forces if the input, as well as output vectors, are adjusted properly according to the production technology. A proper analysis of this matrix will clarify the possible level of competition among the rivals. A new entrant will certainly bring a new set of models which may lead to a comprehensive change in the intensity of rivalry. The power of suppliers & buyers can be analyzed by choosing an appropriate input-output vector. Moreover, the effect of substitutable products can also be explored from such a matrix.
- underlying directions that appear after the Principal Component Analysis which can never be identical for any two given markets. These directions are useful to highlight the way of being more competitive in reference to other rivals.
- competition in a single or multiple types of markets (such as local or global, small or large, etc)
- temporal property of the competitive intensity. Hence, one can evaluate the stability of the market based on the same matrix by observing the statistical significance of the differences.

- the capability of the firm at that particular point of time with the aid of PCA

4. The Prescribed Output Oriented Model:

The output oriented Linear Directional Distance model is mentioned below along with the Variable Return to Scale technology in presence of mix related inefficiencies. The inefficiency of any Decision Making Unit is measured along the first Principal Eigen Vector (E_{1Y}) of the matrix (yy^T).

Output oriented Model
$\max \beta + e'^T S_I + e''^T S_O$ $x\lambda + S_I = x_0$ $y\lambda - \beta E_{1Y} - S_O = y_0$ $\lambda = [\lambda_1 \quad \lambda_2 \cdots \quad \lambda_c]^T$ $\sum_{i=1}^{c} \lambda_i = 1$ $\lambda_i \geq 0$ (11)
Where E_{1Y} is the Eigen Vector of yy^T

4.1 Detection of an Efficient DMU:

In an Input (output) Oriented Model, the DMU under investigation will be termed as a Strongly Efficient Performer if the value of $\lambda_o = 1$ and β along with all slacks remain at zeroes in the following problem.

$$\max \beta + e'^T S_I + e''^T S_O$$

$$x\lambda + S_I = x_0$$

$$y\lambda - \beta E_{1Y} - S_O = y_0$$

$$\sum_{i=1}^{c} \lambda_i = 1 \text{ such that } \lambda_i \geq 0 \qquad (12)$$

A Strongly Efficient DMU neither has any mix inefficiencies attached with the input resources nor is any improvement in the directional distance afforded. The presence of β in a small amount in the above problem opens up the opportunity for the DMU O to improve. This would lead to the following inequality (even if other slacks are zeros):

$$x\lambda = x_0$$

$$y\lambda - \beta E_{1Y} = y_0$$

But, it leads to $y\lambda < y_0$ as $\beta E_{1Y} > 0$. This can never occur for an efficient DMU. Moreover, the emergence of any slack for a particular output will again thrust it towards the mix inefficiency and results in a similar type of inequality.

However, in spite of such conditions, an alternative solution may appear which sets $\beta = 0$ even after satisfying the equality with $\lambda_0 = 1$ or with $\lambda_i \neq 0$ for a few values of i.

$x\lambda = x_0$

$y\lambda = y_0$

$\sum_{i=1}^{c} \lambda_i = 1$

This is also strictly prohibited as per the mandates of efficient DMU. An efficient DMU cannot have any alternative solution. Hence, it is proved that three conditions are important for becoming an efficient DMU.

4.2. DEA Model for Measuring Efficiency Score:

This model is unable to produce efficiency scores like Range Directional Model (RDM^+). A minor modification is made in the objective function to serve the purpose. Considering the output-oriented model (4), the input constraint is summarized to the following expression:

$y\left(\lambda - \beta y^T (yy^T)^{-1} g_y\right) \geq y_0$

$y\left(\lambda - \frac{\beta}{\gamma_1} y^T E_1\right) \geq y_0$ (After inserting the value of g_Y)

In this regard, the new Return to Scale will be

$$e^t\left(\lambda - \frac{\beta}{\gamma_1} y^T E_{1Y}\right) = \sum_{i=1}^{c} \lambda_i - \frac{\beta}{\gamma_1} e^T y^T E_{1Y} = \sum_{i=1}^{c} \lambda_i - \frac{c\beta}{\gamma_1} \bar{y} E_{1Y} \qquad (13)$$

where $\bar{y}$ is the mean vector and $e^T = [1.. \quad 1 \quad 1]$. It implies that the real RTS gets decreased by an amount which is proportional to the projection of the mean vector on the First Principal Component Eigen Vector. It implies that to restrict β within zero to one, βE_{1Y} of the output constraint mentioned in (12) has to be replaced by either $\beta \bar{y} E_{1Y}$ or $\beta \max_i (y_i E_{1Y}) E_{1Y}$. Here, the expression of $\max_i (y_i E_{1Y})$ denotes the largest possible projected length of the output vector obtained on the Eigenvector mentioned before. However, anticipating the presence of worst performers in the sample of firms the later one seems to be better than the former one. Hence, the modified model is given as follows:

Output oriented Model
max $\quad \beta + e'^T S_I + e''^T S_O$
$x\lambda + S_I = x_0$
$y\lambda - \beta \max_i (y_i E_{1Y})\, E_{1Y} - S_O = y_0$
$\lambda = [\lambda_1 \quad \lambda_2 \cdots \quad \lambda_c]^T$
$\sum_{i=1}^{c} \lambda_i = 1 \ \& \ \lambda_i \geq 0$ (14)

4.3. Unit Invariance Property of the Oriented Models:

The non-normalised model does not seem to be unit invariant. The direction vectors such as $E_{1x}\ and\ E_{1y}$ are needed to be computed from analyzing $xx^T and\ yy^T$ respectively (in short non-normalized input and output based MHHI). Let there be a diagonal matrix $\emptyset$ which represents input units and X_i is the numerical value of goods consumed by the i^{th} firm having being supplied by a number of suppliers.

Hence, $xx^T = \sum_{i=1}^{c} \emptyset X_i^T X_i \emptyset^T = \emptyset\left(\sum_{i=1}^{c} X_i^T X_i\right)\emptyset,\ as\ \emptyset^T = \emptyset$

As a result, E_{1x} $(also\ E_{1y})$ will be affected by the dissimilar units mentioned as $\emptyset$ of x (y). To nullify this complexity, a replacement of x_{ij} (y_{ij}) with $\frac{x_{ij}}{\sum_{j=1}^{c} x_{ij}} \left(\frac{y_{ij}}{\sum_{j=1}^{c} y_{ij}}\right)$ is made in the normalised model.

5. Example:

Let there are two line items (outputs) produced from the assemblage of two inputs purchased from a single supplier (Table 1). The production plans of five rivals (from A to E) are elucidated to construct a feasible production possibility set. DMU 5 attains a dominant position owing to its higher volume of sales in the context of both goods. Sellers' concentration in both of these cases is appreciably high (as the measure of $CR2$ in both of these cases is 58% and 57% respectively). Moreover, the HHI values obtained from this example are remarkably high (2299 and 2296) and hence can also be an indicative of highly concentrated market.

<INSERT TABLE 1: DATA SET>

Theoretically, a dominant firm (like DMU 5) assumes a large portion of the HHI and therefore will be responsible for creating a comparatively high non-centred covariance in the market. The non-normalised HHI values of all inputs and outputs are described in the diagonal elements of Table 2. DMU 5 contributes to a bigger portions (100*36/83) % and (100*81/180) % in the non-central variation of two inputs. The combined non-normalised non-centred covariance matrix (with respect to origin) referred in Table 2 is used for deriving all possible Eigenvalues as well as Vectors (Table 3 and Table 4) using SPSS 16. This table keeps vital information about the direction of concentration levels in the buyer and seller market.

<INSERT TABLE 2: **Variance-Covariance Matrix of the Data** >

<INSERT TABLE 3: **COMPOSITION OF EIGEN VECTORS**>

<INSERT TABLE 4: **Variance Explained by Eigen Vectors**>

In Theorem 4 it has been indicated that the first Eigen Vector of Variance-Covariance matrix of inputs and outputs does not possess all positive constituent elements. It can be observed that the third vector is containing it but the degree of explanation of variation is merely 7%. So, the non-oriented model is not computed at this stage (only input and output oriented models are approached).

Table 5 to Table 8 is mentioned to display that the first Eigenvector is the best one for both of these occasions to explain most of the variation. For example, the first Output oriented Eigenvector explains variation up to 96.4% (254.4) out of the total non-centred variation of 263 (83+180) due to two outputs. Directions obtained from Table 5 and Table 7 primarily describe the multi-dimensional concentration of the Buyer and a Seller industry. Table 6 and Table 8 are meant for pointing out the level of variation owing to these Directions (mentioned in Table 5 & Table 7). In other words, the extent of heterogeneity of performance is reflected from these input and output oriented directions. On both occasions, the first Eigenvectors are capable of explaining the highest possible variation and hence afford to describe the major portion of the market concentration. However, to justify whether the market is too much concentrated or not the variance of the output oriented Eigenvalues are computed (which is equal to 0.0902). As per the earlier discussion, in any case the variance should stay within 0.08 and 2 (as the limits are $\left(\frac{m}{c^2}\right) and\ (m)$). Observing the proximity of the derived score it can be inferred that the market is not at all dominated by any particular firm.

To understand the extent of cost leadership of a firm the output-oriented model is run in the framework of Lingo 13. The end result is stated in Table 9.

<INSERT TABLE 5: **COMPOSITION OF EIGEN VECTORS FOR INPUTS**>

<INSERT TABLE 6: **Variance Explained by Eigen Vectors for Inputs**>

<INSERT TABLE 7: **COMPOSITION OF EIGEN VECTORS FOR OUTPUTS**>

<INSERT TABLE 8: **Variance Explained by Eigen Vectors for Outputs**>

<Insert Table 9: Optimal Solution from the Output Oriented Model>

DMU 1, DMU 4 and DMU 5 are found to be efficient owing to their competencies. Conversely, DMU 2 and DMU 3 have to recuperate their processes to produce goods in the stated manner. The improvement will lead them to an advanced competitive position that would neutralize the power of their rivals and revitalize their strength in the market.

This solution is compiled to obtain new peers for the inefficient DMUs. For example, peers prescribed for DMU 2 and DMU 3, in this scenario, are found as (3, 4.25; 3.75, 7.5) and (4, 3; 4.4, 5.8) (shown in Table 10). A non-centred covariance matrix (Table 11) is recomputed with these peers along with the old ones to observe the change in ratio of total variation between outputs and inputs. A sharp increase in this ratio is expectedly noticed from 1.48 $\left(\frac{83+180}{90+87}\right)$ to 2.09 $\left(\frac{103.4+228.9}{90+69}\right)$.

<Insert Table 10: New Data Set>

<Insert Table 11: Variance-Covariance Matrix of the New Data >

This outcome ultimately raises a question that whether this model is capable of distributing the concentration evenly or not. To answer this query an output oriented BCC model is run on the identical input data & the output data projected on the first Eigen vector. Efficiency scores in this case will imply the relative competitive positioning of a firm due to the utilisation of its inputs and selling of its outputs (Table 12).

<Insert Table 12: Relative Competitive Positioning Scores >

The set of peers emerged from this optimization are totally identical to those ones which were predicted earlier by the prescribed Linear DDF. Hence, the way of consuming resources and selling of outputs to attain a superior competitive positioning can be suggested.

6. Conclusion:

The state of art of this research is indicative of the directions which are primarily able to determine the status of a firm in the context of the Cost Leadership Strategy. In absence of price or cost data the volume of inputs or outputs can be the next authentic means to elucidate the competitive position. Two independent directions are obtained from the assessment of input and output vectors separately. These directions seem to be similar to the cost per unit or revenue per unit. A cost leader would certainly result in either least possible projection on the first Principal Eigenvector obtained from un-centred input based covariance matrix or possess a largest projected length on the first Principal Eigenvector of the un-centred output based covariance matrix. However, the model has a severe drawback of not-withstanding the negative input or output data. Under VRS concepts, a DDF, although, does not offer any restriction on the negative data, however, the translation of a suitable amount to the input or output data may cause changes in the orientation of the Eigenvectors.

Lastly, the method is reliable as well as valid too. Reliability is a measure of assessing the extent of variation owing to the presence of random error. A higher variation will cause a large deviation from the true value of the quantity to be measured. Hence, a reliable measurement of the quantity will always keep the observations close to the true measurements. The aim of this paper is to extend the concept of HHI so as to utilize it for the sake of line items and substitute products. HHI has been pervasively applied for measuring competition among banks, airline companies and for many other sectors. Apart from Concentration Ratios this is the only reliable tool to serve the purpose. Hence, the proposed analysis will also be reliable.

Validity refers to ability of the tool to measure a quantity what it wants to measure. The model prescribed here invokes a fundamental assumption that the competition should be thought of like a vector instead of a scalar. HHI on a particular direction is assumed here to be the indicator of it. Referring to the concepts of Porter (1980) & his lectures on Competitive Strategy, it is added here that "a strategy is the way to remain unique in any competition". The author summarised that the attainment of a sustained competitive advantage is possible when a firm selects a unique way of competing in the market rather than choosing to be the best in the same field by ameliorating its operations. Hence, there can be a large number of ways to compete in the market and firm has to choose that path which suites itself in a best manner. MHHI provides plenty of such directions on which a firm can excel. However, the Additive model of Chamber et al (1998) selects only the first Eigen-vector of MHHI which is associated with the largest Eigen-value. This direction will reflect the largest possible variation among the firms. Hence, the calculation of HHI on this direction will display the might of certain firms on it. Thus, a firm (which is being dominated) can thrive along this direction to counterpoise the high level of market concentration.

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APPENDIX 1 (Properties of DDF):

The contemporary researches revealed that any *DDF* must possess subsequent properties:

a. Translation: $D\left(x - \alpha g_x, y + \alpha g_y, g_x, g_y\right) = D\left(x, y, -g_x, g_y\right) + \alpha$

b. g-Homogeneity of degree is minus one: $D(x, y, \alpha g) = \alpha^{-1}\, D(x, y, g)$

c. It is monotonic for Good inputs: If $x^{'} \geq x, then\; D(x', y, g) \geq D(x, y, g)$

d. It is monotonic for Good outputs: If $y^{'} \geq y, then\; D(x, y, g) \geq D(x, y', g)$

e. It is concave.

f. It is non-negative: $D(x, y, g) \geq 0$

So, the subsequent proofs are addressed below in order to verify those imperative properties. $x\; and\; y$ symbolize the input and output vectors which are permissible to the technology *T*. Here, g denotes a feasible direction vector.

a. **Proof of Translation:**

Let there exists a DDF $D(x - g_x\alpha, y)$ which is involved in the same technology then:

$D(x - g_x\alpha, y)$

$= \{(x - g_x\alpha - g_x\beta, y)\epsilon L(y)\}$

$= \{(x - g_x(\alpha + \beta), y)\epsilon L(y)\}$

$= \{(x - g_x\beta' xA, y)\epsilon L(y)\} + \alpha$ when $\beta^{'} = (\alpha + \beta)$

$= D(x, y, g_x) + \alpha$

Hence, a linear shift of α along the direction vector leads to a displacement of α from the original position.

b. **Proof of the *g*-Homogeneity:**

The proposed DDF is designed to follow the condition

$D\left(x, y, g_x\alpha, g_y\alpha\right) = \left\{\left(x - g_x\alpha\beta, y + g_y\alpha\beta\right)\epsilon T\right\}$

Or $D\left(x, y, g_x, g_y\right) = \left\{\left(x - g_x\beta', y + g_y\beta'\right)\epsilon T\right\}$ for $\beta' = \alpha\beta$

The former expression delineates the distance in terms of β whereas the later one measures it through β'. It implies that $D\left(x, y, g_x\alpha, g_y\alpha\right) = \frac{1}{\alpha} D\left(x, y, g_x, g_y\right)$

c. It is monotonic for Good inputs:

If $x^{'} \geq x$, such that $x^{'} = x + \alpha g_x$ then from theorem 3 it can be stated that $D(x', y, g) = D(x, y, g) + \alpha$. Due to $\alpha > 0$ the distance function $D(x', y, g)$ is definitely larger than $D(x, y, g)$.

d. **It is monotonic for Good Outputs:**

Similar steps required for displaying the property 3 will be sufficient to prove it.

e. Proof of Concavity:

To prove convexity the above *DDF* with three different input combinations such as $D(x', y, g_x, 0), D(x'', y, g_x, 0)$ *and* $D(x, y, g_x, 0)$ at $x^{'}, x^{''}$ *and* x are cited. It is also assumed that these inputs have a convex relation $\vartheta_1 x^{'} + (1 - \vartheta_1)x^{''} = x$ where $x^{'} > x^{''}$ and $1 \geq \vartheta_1 \geq 0$. Moreover, $x = x^{'} - t_1 g_x$ and $x = x^{''} + t_2 g_x$. Since *T* is a convex set, so, x can be a feasible input for the production volume of *y*. Then, the subsequent inequalities are issued due to the general properties of *DDF*:

$D(x', y, g_x, 0) > D(x, y, g_x, 0) > D(x^{''}, y, g_x, 0)$

Then, $\vartheta_1 D(x^{'}, y, g_x, 0) + (1 - \vartheta_1) D(x^{''}, y, g_x, 0)$

$\leq D(x, y, g_x, 0) + \vartheta_1 t_1 = D(x + \vartheta_1 t_1 g_x, y, g_x, 0)$

But, due to $1 \geq \vartheta_1 \geq 0$, the relationship $x^{'} > x + \vartheta_1 t_1 g_x > x$ can be obtained. This inequality entails that there exists a point $x + \vartheta_1 t_1 g_x$ at which the *DDF* score is more than the convex combination of two *DDFs* measured at $x^{'}$ *and* $x^{'}{}'$. Hence, it is concluded that *DDF* is concave in nature.

f. **Proof of non-negativity:**

Consider a DDF which is mentioned as $D(x, y, g_x, 0) = \{(x - \beta g_x, y) \epsilon T\}$

It implies that $D(x, y, g_x, 0) = \{x(I - g_x \beta), y \epsilon T\}$

Now, according to the rule, a movement along the direction vector will result in the lesser volume of input consumption which again entails that $x(I - g_x \beta) \leq x$ or $(x g_x \beta \geq 0)$. But, since $x \geq 0$ *and* $g_x \geq 0$, so, they lead to a conclusion that $\beta \geq 0$.

APPENDIX 2 (Extension of Lerner's Theory & Multi-Dimensional HHI):

Let there be two rivals competing in a market with the two types of products having market prices of p_1 *and* p_2. y_{11} *and* y_{21} (y_{12} *and* y_{22}) are the volumes of sold quantities of 1st and 2nd product by the 1st producer (2nd producer). Y_1 *and* Y_2 are the total sold quantities of 1st and 2nd product, or $Y_1 = \sum_{i=1}^{2} y_{i1}$, $Y_2 = \sum_{i=1}^{2} y_{i2}$.

The matrix $z = \begin{bmatrix} \frac{y_{11}}{Y_1} & \frac{y_{12}}{Y_1} \\ \frac{y_{21}}{Y_2} & \frac{y_{22}}{Y_2} \end{bmatrix}$ corresponds to the representation of market shares of two rivals.

According to the derivation of Bain, the condition of profit maximization for the first firm is formulated through the marginal cost function and the function of price as below:

$\frac{\partial \pi_{11}}{\partial y_{11}} = p_1 + y_{11} \frac{\partial p_1}{\partial y_{11}} - \frac{\partial C(y_{11})}{\partial y_{11}} = 0$ ....(1)

To permit the cross elasticity effects, prices are supposed to be functions of the total sold quantities of the two products, hence, $p_1 = f(Y_1, Y_2)$. The partial derivation in the above equation can then be replaced by the subsequent terms:

$$\frac{\partial p_1}{\partial y_{11}} = \frac{\partial p_1}{\partial Y_1}\frac{\partial Y_1}{\partial y_{11}} + \frac{\partial p_1}{\partial Y_2}\frac{\partial Y_2}{\partial Y_1}\frac{\partial Y_1}{\partial y_{11}}$$

$$= \frac{p_1 e_{11}}{Y_1} + MRS_{21}\frac{p_1 e_{12}}{Y_1} \text{ ....(2)}$$

Here, $e_{11}$is the measure the change in the demand due to the changes made in the price of product 1. On the other hand, MRS_{21} is the marginal rate of substitution of product 2 due to the changes occurred in the sales volume of the product 1.

$$\frac{\partial \pi_{11}}{\partial y_{11}} = p_1 + y_{11}\left(\frac{p_1 e_{11}}{Y_1} + MRS_{21}\frac{p_1 e_{12}}{Y_1}\right) - \frac{\partial C(y_{11})}{\partial Y_{11}} = 0 \text{ ....(3)}$$

After rearranging and multiplying (3) with y_{11} on both sides the market power of the 1st producer, I_1, will be obtained:

$$I_1 = \frac{\left(p_1 y_{11} - \frac{\partial C(y_{11})}{\partial y_{11}} y_{11}\right)}{p_1 Y_1} = -\frac{{y_{11}}^2}{{Y_1}^2}(e_{11} + MRS_{21}e_{12}) = -\frac{{y_{11}}^2}{{Y_1}^2}t_{11} \text{ .....(4)}$$

Similarly, the index due to the second product is given as

$$I_3 = -\frac{{y_{21}}^2}{{Y_2}^2}(e_{22} + MRS_{12}e_{21}) = -\frac{{y_{21}}^2}{{Y_2}^2}t_{22} \text{ ...(5)}$$

On the other hand, the product of (4) and x_{12} will result in a new index I_2:

$$I_2 = \frac{\left(p_1 y_{21} - \frac{\partial C(y_{11})}{\partial y_{11}} y_{21}\right)}{p_1 Y_2} = -\frac{y_{11}y_{21}}{Y_2 Y_1}(e_{11} + MRS_{21}e_{12}) = -\frac{y_{11}y_{21}}{Y_2 Y_1}t_{11} \text{ .....(6)}$$

I_2 is supposedly a reflection of the market power of the 1st producer due to the complete substitution of the 2nd one with 1st product. In other words, I_2 describes the market power if the 2nd product was completely substituted by 1st product. Terms like $t_{11}\ or\ t_{22}$ are involved here to mention the effect of change of sales volume on the price of a product. In other words, both the price elasticity and cross elasticity are assumed to be feasible for this occasion. Hence, the matrix form (*SS*) of the market power due to 1st producer is mentioned as follows:

$$SS = \begin{bmatrix} I_1 & I_2 \\ I_2 & I_3 \end{bmatrix} = -\begin{bmatrix} \frac{{y_{11}}^2}{{Y_1}^2}t_{11} & \frac{y_{11}y_{21}}{Y_2 Y_1}t_{11} \\ \frac{y_{11}y_{21}}{Y_2 Y_1}t_{11} & \frac{{y_{21}}^2}{{Y_2}^2}t_{22} \end{bmatrix} \text{ ......(7)}$$

For the sake of simplicity, the market power of the 1st producer is expressed in terms of the matrix S_1 (shown below) which is obtained after summarizing the contribution of the same producer in the domain of concentration owing to the sales pattern of the two products within the market.

$$S_1 = -\begin{bmatrix} \frac{{y_{11}}^2}{{Y_1}^2} & \frac{y_{11}y_{21}}{Y_2Y_1} \\ \frac{y_{11}y_{21}}{Y_2Y_1} & \frac{{y_{21}}^2}{{Y_2}^2} \end{bmatrix} \text{....(8)}$$

$\frac{{y_{11}}^2}{{Y_1}^2}$ and $\frac{{y_{21}}^2}{{Y_2}^2}$ are the squared market share due to presence of the 1st producer in the business of two substitutable products. $\frac{y_{11}y_{21}}{Y_2Y_1}$ is the *product of share* caused by the same producer developed from two goods. In a similar fashion, the augmentation of market power from the second firm is cited as $S_2 = -\begin{bmatrix} \frac{{y_{12}}^2}{{Y_1}^2} & \frac{y_{12}y_{22}}{Y_2Y_1} \\ \frac{y_{12}y_{22}}{Y_2Y_1} & \frac{{y_{22}}^2}{{Y_2}^2} \end{bmatrix}$ .....(9)

The multi-dimensional concentration emerged due to total market power is referred as

$$S_1 + S_2 = -\begin{bmatrix} \frac{{y_{11}}^2}{{Y_1}^2} + \frac{{y_{12}}^2}{{Y_1}^2} & \frac{y_{11}y_{21}}{Y_2Y_1} + \frac{y_{12}y_{22}}{Y_2Y_1} \\ \frac{y_{11}y_{21}}{Y_2Y_1} + \frac{y_{12}y_{22}}{Y_2Y_1} & \frac{{y_{21}}^2}{{Y_2}^2} + \frac{{y_{22}}^2}{{Y_2}^2} \end{bmatrix} = MHHI = z^T z \text{ .....(10)}$$

Table 1: Data Set

DMU	Input 1	Input 2	Output 1	Output 2
1	2	4	3	7
2	3	6	2	4
3	4	3	3	5
4	5	1	5	3
5	6	5	6	9

Table 2: Variance-Covariance Matrix of the Data

	I1	I2	O1	O2
I1	90	73	-85	-115
I2	73	87	-68	-115
O1	-85	-68	83	113
O2	-115	-115	113	180

Table 3: COMPOSITION OF EIGEN VECTORS

	E1	E2	E3	E4
I1	-4.282	-1.123	.598	.211
I2	4.129	1.149	.044	.268
O3	6.236	-.571	.879	-.086
O4	-4.072	1.471	.763	-.082

Table 4: Variance Explained by Eigen Vectors

Components	Total	% of Variance	Cumulative %
E1	408.87	92.924	92.924
E2	22.83	5.188	98.112
E3	7.72	1.754	99.866
E4	0.59	.134	100.000

Table 5: COMPOSITION OF EIGEN VECTORS FOR INPUTS

	E1	E2		E1	E2
I1	0.957	-0.29		0.719	0.695
I2	0.954	0.30		0.695	-0.719

Table 6: Variance Explained by Eigen Vectors for Inputs

Component	Extraction Sums of Squared Loadings		
	Total	% of Variance	Cumulative %
E1	161.5	91.252	91.252
E2	15.5	8.748	100.000

Table 7: COMPOSITION OF EIGEN VECTORS FOR OUTPUTS

	E1	E2		E1	E2
O1	0.964	0.268		0.550	0.835
O2	0.993	-0.120		0.835	-0.550

Table 8: Variance Explained by Eigen Vectors for Outputs

Component	Extraction Sums of Squared Loadings		
	Total	% of Variance	Cumulative %
E1	254.47	96.756	96.756
E2	8.53	3.244	100.000

Table 9: Optimal Solution of the Output Oriented Model

	A		B		C		D		E	
Variable	Value	Reduced Cost	Value	Reduced Cost	Value	Reduced Cost	Value	Reduced Cost	Value	Reduced Cost
β	0.000	0.000	0.294	0.000	0.089	0.000	0.000	0.000	0.000	0.000
S1	0.000	0.055	0.000	0.126	0.000	0.015	0.000	0.000	0.000	0.000
S2	0.000	0.000	1.750	0.000	0.000	0.162	0.000	0.166	0.000	0.000
S3	0.000	0.000	0.000	0.168	0.873	0.000	0.000	0.000	0.000	0.000
S4	0.000	0.111	0.843	0.000	0.000	0.111	0.000	0.111	0.000	0.111
λ_1	1.000	0.000	0.750	0.000	0.400	0.000	0.000	0.055	0.000	0.221
λ_2	0.000	0.388	0.000	0.294	0.000	0.672	0.000	0.719	0.000	0.554
λ_3	0.000	0.332	0.000	0.252	0.000	0.089	0.000	0.111	0.000	0.443
λ_4	0.000	0.608	0.000	0.042	0.400	0.000	1.000	0.000	0.000	0.664
λ_5	0.000	0.000	0.250	0.000	0.200	0.000	0.000	0.000	1.000	0.000
Row	Slack/ Surplus	Dual Price	Slack/ Surplus	Dual Price	Slack/ Surplus	Dual Price	Slack/ Surplus	Dual Price	Slack/ Surplus	Dual Price
1	0.000	0.055	0.000	0.126	0.000	0.015	0.000	0.000	0.000	0.000
2	0.000	0.000	0.000	0.000	0.000	0.162	0.000	0.166	0.000	0.000
3	0.000	0.000	0.000	-0.168	0.000	0.000	0.000	0.000	0.000	0.000
4	0.000	-0.111	0.000	0.000	0.000	-0.111	0.000	-0.111	0.000	-0.111
5	0.000	0.664	0.000	0.252	0.000	0.096	0.000	0.166	0.000	0.996

Table 10: New Data Set

DMU	Input 1	Input 2	Output 1	Output 2
1	2	4	3	7
2	3	4.25	3.75	7.5
3	4	3	4.4	5.8
4	5	1	5	3
5	6	5	6	9

Table 11: Variance-Covariance Matrix of the New Data

90	67.75	95.85	128.7
67.75	69.0625	76.1375	125.275
95.85	76.1375	103.4225	143.645
128.7	125.275	143.645	228.89

Table 12: Relative Competitive Positioning Scores

	A		B		C		D		E	
Variable	Value	Reduced Cost	Value	Reduced Cost	Value	Reduced Cost	Value	Reduced Cost	Value	Reduced Cost
β	1.000	0.000	1.875	0.000	1.247	0.000	1.000	0.000	1.000	0.000
S1	0.000	0.500	0.000	0.187	0.000	0.088	0.000	0.143	0.000	0.077
S2	0.000	0.000	1.750	0.000	0.000	0.217	0.000	0.285	0.000	0.000
S3	0.000	0.133	0.000	0.225	0.000	0.172	0.000	0.190	0.000	0.092
λ_1	1.000	0.000	0.750	0.000	0.400	0.000	0.000	0.000	0.000	0.000
λ_2	0.000	0.908	0.000	0.875	0.000	1.046	0.000	1.294	0.000	0.359
λ_3	0.000	1.223	0.000	0.750	0.000	0.247	0.000	0.319	0.000	0.308
λ_4	0.000	1.799	0.000	1.065	0.400	0.000	1.000	0.000	0.000	0.437
λ_5	0.000	1.557	0.250	0.000	0.200	0.000	0.000	0.225	1.000	0.000
Row	Slack/ Surplus	Dual Price	Slack/ Surplus	Dual Price	Slack/ Surplus	Dual Price	Slack/ Surplus	Dual Price	Slack/ Surplus	Dual Price
1	0.000	0.500	0.000	0.187	0.000	0.088	0.000	0.143	0.000	0.077
2	0.000	0.000	0.000	0.000	0.000	0.217	0.000	0.285	0.000	0.000
3	0.000	-0.133	0.000	-0.225	0.000	-0.172	0.000	-0.190	0.000	-0.092
4	0.000	0.000	0.000	1.314	0.000	0.244	0.000	0.000	0.000	0.540