The Quantified Reflection Calculus as a Modal Logic

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1 Introduction

The Quantified Reflection Calculus with one modality, denoted by $\mathsf{QRC_1}$ and introduced in [1], is a strictly positive quantified modal logic inspired by the unimodal fragment of the Reflection Calculus, $\mathsf{RC_1}$ [4, 3]. The quantified strictly positive language consists of a *verum* constant and relation symbols as atomic formulas, with the only available connectives being the conjunction, the diamond, and the universal quantifier. $\mathsf{QRC_1}$ statements are assertions of the form $\varphi \vdash \psi$ where φ and ψ are in this strictly positive language.

$\mathsf{QRC_1}$ was born out of the wish for a nice quantified provability logic for theories of arithmetic such as Peano Arithmetic, even though Vardanyan [6] showed that this is in general impossible. In fact, the full quantified provability logic of PA is Π^0_2-complete, and thus not recursively axiomatizable, let alone decidable. However, restricting the language to the strictly positive fragment is a viable solution, as shown by the authors in [2].

The main results described here are the Kripke soundness and completeness for $\mathsf{QRC_1}$, as well as a work-in-progress description of the polymodal extension QRC_N. We obtain constant domain completeness for $\mathsf{QRC_1}$ as well as the finite model property, implying its decidability. Together with the arithmetical results described in [2], this means that $\mathsf{QRC_1}$ is a very nice provability logic indeed.

There is an ongoing formalization[1] of $\mathsf{QRC_1}$ in the Coq Proof Assistant [5].

2 $\mathsf{QRC_1}$

$\mathsf{QRC_1}$ talks about quantified strictly positive formulas as described above, where the terms are either variables or constant symbols (no function symbols).

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[1] https://gitlab.com/ana-borges/QRC1-Coq

The free variables of a formula φ are defined as usual, and denoted by $\mathrm{fv}(\varphi)$. The expression $\varphi[x\leftarrow t]$ denotes the formula φ with all free occurrences of the variable x simultaneously replaced by the term t. We say that t is free for x in φ if no occurrence of a free variable in t becomes bound in $\varphi[x\leftarrow t]$.

The axioms and rules of $\mathsf{QRC_1}$ are listed in the following definition from [2].

Definition 2.1 ($\mathsf{QRC_1}$, [2])**.** Let φ, ψ, and χ be any quantified strictly positive formulas. The axioms and rules of $\mathsf{QRC_1}$ are the following:

(i) $\varphi \vdash \top$ and $\varphi \vdash \varphi$;

(ii) $\varphi \wedge \psi \vdash \varphi$ and $\varphi \wedge \psi \vdash \psi$;

(iii) if $\varphi \vdash \psi$ and $\varphi \vdash \chi$, then $\varphi \vdash \psi \wedge \chi$;

(iv) if $\varphi \vdash \psi$ and $\psi \vdash \chi$, then $\varphi \vdash \chi$;

(v) if $\varphi \vdash \psi$, then $\Diamond\varphi \vdash \Diamond\psi$;

(vi) $\Diamond\Diamond\varphi \vdash \Diamond\varphi$;

(vii) if $\varphi \vdash \psi$, then $\varphi \vdash \forall x\, \psi$ ($x \notin \mathrm{fv}(\varphi)$);

(viii) if $\varphi[x\leftarrow t] \vdash \psi$ then $\forall x\, \varphi \vdash \psi$ (t free for x in φ);

(ix) if $\varphi \vdash \psi$, then $\varphi[x\leftarrow t] \vdash \psi[x\leftarrow t]$ (t free for x in φ and ψ);

(x) if $\varphi[x\leftarrow c] \vdash \psi[x\leftarrow c]$, then $\varphi \vdash \psi$ (c not in φ nor ψ).

If $\varphi \vdash \psi$, we say that ψ follows from φ in $\mathsf{QRC_1}$.

We observe that our axioms do not include universal quantifier elimination. However, this and various other rules are readily available via the following easy lemma.

Lemma 2.2. The following are theorems (or derivable rules) of $\mathsf{QRC_1}$:

(i) $\forall x\, \forall y\, \varphi \vdash \forall y\, \forall x\, \varphi$;

(ii) $\forall x\, \varphi \vdash \varphi[x\leftarrow t]$ (t free for x in φ);

(iii) $\Diamond \forall x\, \varphi \vdash \forall x\, \Diamond\varphi$;

(iv) $\forall x\, \varphi \vdash \forall y\, \varphi[x\leftarrow y]$ (y free for x in φ and $y \notin \mathrm{fv}(\varphi)$);

(v) if $\varphi \vdash \psi$, then $\varphi \vdash \psi[x\leftarrow t]$ (x not free in φ and t free for x in ψ);

(vi) if $\varphi \vdash \psi[x\leftarrow c]$, then $\varphi \vdash \forall x\, \psi$ (x not free in φ and c not in φ nor ψ).

The Kripke semantics for $\mathsf{QRC_1}$ is a simple generalization of Kripke semantics for propositional modal logics. For simplicity, we restrict ourselves to constant domain models.

Definition 2.3. An *adequate model* $\mathcal{M}$ in a signature Σ is a tuple $\langle W, R, M, I, \{J_w\}_{w\in W}\rangle$ where:

- W is a non-empty set (the set of worlds, where individual worlds are referred to as w, u, v, etc);

- R is a transitive binary relation on W (the accessibility relation);
- M is a finite set (the domain of the model, whose elements are referred to as d, d_0, d_1, etc);
- the interpretation I assigns an element of the domain M to each constant c in the signature, written c^I; and
- for each $w \in W$, the interpretation J_w assigns a set of tuples $S^{J_w} \subseteq \wp(M^n)$ to each n-ary relation symbol S in the signature.

A model is said to be *finite* is the set of worlds W is finite.

We use assignments to define truth at a world in a first-order model. An assignment g is a function assigning a member of the domain M to each variable in the language.

Two assignments g and h are x-*alternative*, written $g \sim_x h$, if they coincide on all variables other than x. An assignment g is extended to terms by defining $g(c) := c^I$ for any constant c.

We now define satisfaction at a world.

Definition 2.4. Let $\mathcal{M} = \langle W, R, M, I, \{J_w\}_{w \in W} \rangle$ be an adequate model, and let $w \in W$ be a world, g be an assignment, S be an n-ary relation symbol, and φ, ψ be $\mathsf{QRC_1}$ formulas.

We define $\mathcal{M}, w \Vdash^g \varphi$ (φ is true at w under g) by induction on φ as follows.

- $\mathcal{M}, w \Vdash^g \top$;
- $\mathcal{M}, w \Vdash^g S(t_0, \ldots, t_{n-1})$ iff $\langle g(t_0), \ldots, g(t_{n-1}) \rangle \in S^{J_w}$;
- $\mathcal{M}, w \Vdash^g \varphi \wedge \psi$ iff both $\mathcal{M}, w \Vdash^g \varphi$ and $\mathcal{M}, w \Vdash^g \psi$;
- $\mathcal{M}, w \Vdash^g \Diamond\varphi$ iff there is a $v \in W$ such that wRv and $\mathcal{M}, v \Vdash^g \varphi$;
- $\mathcal{M}, w \Vdash^g \forall x\, \varphi$ iff for all assignments h such that $h \sim_x g$, we have $\mathcal{M}, w \Vdash^h \varphi$.

The main results on $\mathsf{QRC_1}$ are as follows.

Theorem 2.5 (Soundness for $\mathsf{QRC_1}$, [1])**.** If $\varphi \vdash \psi$, then for any adequate model $\mathcal{M}$, for any world $w \in W$, and for any assignment g:

$$\mathcal{M}, w \Vdash^g \varphi \implies \mathcal{M}, w \Vdash^g \psi.$$

Theorem 2.6 (Completeness for $\mathsf{QRC_1}$, [2])**.** Let φ, ψ formulas in the language of $\mathsf{QRC_1}$. If $\varphi \nvdash \psi$, then there is an adequate, finite, and irreflexive model $\mathcal{M}$, a world $w \in W$, and an assignment g such that:

$$\mathcal{M}, w \Vdash^g \varphi \quad \text{and} \quad \mathcal{M}, w \nVdash^g \psi.$$

Since we have the finite model property, this completeness result implies the decidability of $\mathsf{QRC_1}$ by Post's Theorem.

3 QRC_N

QRC_1 was inspired by the full Reflection Calculus, RC, and as such it is natural to think of a polymodal version for QRC_1, namely QRC_N, with a fixed but arbitrary N. The language is the same as the language of QRC_1, except instead of one diamond, we include N diamonds $\langle 0\rangle, \langle 1\rangle, \ldots, \langle N-1\rangle$.

Definition 3.1 (QRC_N)**.** Let φ, ψ, and χ be formulas in the quantified and polymodal strictly positive language. Let $n, m < N$ be natural numbers. The axioms and rules of QRC_N are the following:

(i) $\varphi \vdash \top$ and $\varphi \vdash \varphi$;

(ii) $\varphi \wedge \psi \vdash \varphi$ and $\varphi \wedge \psi \vdash \psi$;

(iii) if $\varphi \vdash \psi$ and $\varphi \vdash \chi$, then $\varphi \vdash \psi \wedge \chi$;

(iv) if $\varphi \vdash \psi$ and $\psi \vdash \chi$, then $\varphi \vdash \chi$;

(v) if $\varphi \vdash \psi$, then $\langle n\rangle\varphi \vdash \langle n\rangle\psi$;

(vi) $\langle n\rangle\langle n\rangle\varphi \vdash \langle n\rangle\varphi$;

(vii) $\langle n\rangle\varphi \vdash \langle m\rangle\varphi$, with $m < n$;

(viii) $\langle n\rangle\varphi \wedge \forall \boldsymbol{x}\, \langle m\rangle\psi \vdash \langle n\rangle(\varphi \wedge \forall \boldsymbol{x}\, \langle m\rangle\psi)$, with $m < n$;

(ix) if $\varphi \vdash \psi$, then $\varphi \vdash \forall x\, \psi$ ($x \notin \mathrm{fv}(\varphi)$);

(x) if $\varphi[x \leftarrow t] \vdash \psi$ then $\forall x\, \varphi \vdash \psi$ (t free for x in φ);

(xi) if $\varphi \vdash \psi$, then $\varphi[x \leftarrow t] \vdash \psi[x \leftarrow t]$ (t free for x in φ and ψ);

(xii) if $\varphi[x \leftarrow c] \vdash \psi[x \leftarrow c]$, then $\varphi \vdash \psi$ (c not in φ nor ψ).

In Axiom 3.1.(viii), the notation $\forall \boldsymbol{x}$ is meant to describe a (possibly empty) finite string of quantification over different variables that could alternatively have been written as $\forall x_0 \cdots \forall x_{k-1}$.

Kripke models are defined similarly to the ones for QRC_1, except with a relation R_n for each $n < N$. These relations must have the following properties:

- poly-transitive: for each $n, m < N$, if $wR_n u$ and $uR_m v$, then $wR_{\min\{n,m\}} v$;
- monotone: for each $m < n < N$, if $wR_n u$, then $wR_m u$; and
- poly-Euclidean: for each $m < n < N$, if $wR_n u$ and $wR_m v$, then $uR_m v$.

QRC_N is sound for such Kripke models, but the completeness is still work in progress. The main difficulty is with Axiom 3.1.(viii), which requires the relations to be poly-Euclidean. This requirement means that tree-like models are not enough, and the ideas of the completeness proof for QRC_1 can not be trivially reused. This axiom represents an important arithmetical fact (observing that $\langle m\rangle\psi$ represents a Π^0_m sentence in the arithmetical interpretation, and so $\forall \boldsymbol{x}\, \langle m\rangle\psi$ is still Π^0_m), so it is expected that it will be needed to prove the arithmetical completeness of QRC_N, our ultimate goal.

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