No-Three-in-Line Problem and Parafermions

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Abstract

The no-three-in-line problem in discrete geometry asks how many points can be placed in the $n \times n$ grid so that no three points lie on the same line. The problem concerns lines of all slopes, not only those aligned with the grid. It was introduced by Henry Dudeney in 1917. Peter Brass, William Moser and János Pach call it "one of the oldest and most extensively studied geometric questions concerning lattice points".

This number is at most $2n$, since if $2n+1$ points are placed in the grid, then by the pigeonhole principle some row and some column will contain three points. Although the problem can be solved with $2n$ points for every n up to 46, it is conjectured that fewer than $2n$ points are possible for sufficiently large values of n. The best solutions that are known to work for arbitrarily large values of n place slightly fewer than $3n/2$ points.

In this paper the reformulation of the no-three-in-line problem using parafermions is given, which allows to get a better lower bound.

1 Introduction

The no-three-in-line problem is to find the maximum number of points that can be placed in the $n \times n$ grid so that no three points lie on a line. This celebrated century old problem that was posed by Henry Dudeney [1] is still open. For some recent developments, see [2, 3, 4] and references therein.

The no-three-in-line problem was extended to the General Position Subset Selection Problem [4]. Here, for a given set of points in the plane one aims to determine a largest subset of points in general position (finite sets of points with no three in line are said to be in general position). In [4] it was also proved, among other results, that the problem is NP- and APX-hard (the set of NP optimization problems that allow polynomial-time approximation algorithms with approximation ratio bounded by a constant).

While it is unknown for larger n whether the upper bound $2n$ is achievable, there are several constructions where the size of the set is a smaller multiple of n. The earliest of these is due to Paul Erdős [5], and uses the modular parabola, consisting of the points $(i, i^2) \mod p$ (taking p to be the largest prime before n yields $n - o(n)$ points in general position). The best known general construction is due to [6], where it places points on a hyperbola $xy = k \mod p$ with a prime p slightly smaller than $n/2$, and yields $3n/2 - o(n)$ points in general position.

Various upper bound to the problem had also been conjectured. For instance, it is conjectured that (see [7, 8]) the number of points that can be placed in the $n \times n$ grid so that no three are collinear has the optimal solution cn with $c = \pi/\sqrt{3} \approx 1.8137$.

In this paper the reformulation of the no-three-in-line problem using parafermions is given, which allows to get a better lower bound. The introduction of the parafermions [9] in the context of statistical models and conformal field theory is perhaps one of the most significant conceptual advances in modern theoretical physics. Parafermion fields have fractional conformal dimension and are not required to be local to each other, the order of their mutual singularity can be any real number. Parafermionic algebras can be seen as a generalization of standard conformal chiral algebras (vertex algebras in mathematical literature) to the case of nonlocal fields. Ideas from [10, 11, 12] are also notable.

It is worthy to mention the eight queens puzzle, on placing points on the grid with no two on the same row, column, or diagonal. It was first posed in the mid-19th century. In the modern era, it is often used as an example problem for various computer programming techniques. Although the exact number of solutions is only known for $n \leq 27$, the asymptotic growth rate of the number of solutions is approximately $(0.143n)^n$.

2 Statistical Model

To the $n \times n$ grid, which we denote as S_n, there corresponds the parafermion algebra $A(S_n)$ defined as follows (see also [11]).

Let V be the vertex set of S_n. Since there is only one straight line through any two points, for any $v \in V$ and any $w \neq v \in V$ we get the straight line in standard form $ax + by + c = 0$. Hence, for a fixed $v \in V$ the pair a, b completely encodes the straight line, passing through w. Let L_v be the set of all such pairs a, b for a fixed $v \in V$. The algebra $A(S_n)$ is generated by

$$\psi_v = \phi_v \prod_{i \in L_v} \theta^i, v \in V,$$

where ϕ_v is a Z_2 parafermion (i.e. $(\phi_v)^2 = 0$) and θ^i is a Z_3 parafermion (i.e. $(\theta^i)^3 = 0$).

In other words, we have constructed a statistical model in which the space of configurations is the set of arrangements of particles ψ_v such that at each vertex at most one particle is located and three particles cannot be located together if they are collinear. Hence, a configuration with the biggest number of arranged particles gives the answer to the no-three-in-line problem.

Remark 1. *Does a handleable deformation of $A(S_n)$ like in* [10, 12] *exist?*

Remark 2. *Let $l(n)$ be the number of lines through at least two points of the $n \times n$ grid. For all $n \geq 2$, $l(n) = \frac{9n^4}{4\pi^2} + O(n^3 \log n)$, see* [13]*. Note that from a line with (exactly) k grid point we can form $\binom{k}{m}$ different subsets of m collinear points ($k \geq m$).*

Remark 3. *The Szemerédi-Trotter theorem states that for given m points in the Euclidean plane and an integer $k \geq 2$, the number of lines which pass through at least k of the points is*

$$O(\frac{m^2}{k^3} + \frac{m}{k}).$$

Remark 4. *The Beck's theorem asserts the existence of positive constants C, K such that given any m points in the Euclidean plane, at least one of the following statements is true:*

1) There is a line which contains at least m/C of the points.

2) There exist at least n^2/K lines, each of which contains at least two of the points.

In József Beck's original argument, C is 100 and K is an unspecified constant. It is not known what the optimal values of C and K are.

Remark 5. *The Sylvester–Gallai theorem states that every finite set of points in the Euclidean plane has a line that passes through exactly two of the points or a line that passes through all.*

Remark 6. *Z_3 parafermions can be used to produce Fibonacci anyons (Temperley–Lieb algebra, RSOS model; that's it: there exists a fermionic description), laying a path towards universal topological quantum computation* [14].

Remark 7. *Note the 3-rule, which says that particles prefer to be mostly 3 sites apart* [15][16].

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