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Bottom-up Sequentialization of Unit-Free MALL Proof Nets

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We propose a new proof of sequentialization for the proof nets of unit-free multiplicative-additive linear logic of Hughes & Van Glabbeek [HG05]. This is done by adapting a method from unit-free multiplicative linear logic [Lau13], showing the robustness of this approach.

1 Introduction

Proof nets are a major contribution from linear logic (LL) [Gir87]. They represent proofs, no longer as trees, but as more general graphs, identifying proofs up to rule permutations [HG16]. This yields more canonical objects, on which results such as cut elimination become easier to prove. A difficult theorem to prove on proof nets is that they indeed correspond to proof trees. The hard part is building a tree from a net, a process called sequentialization.

We study here cut-free proof nets for the unit-free multiplicative-additive fragment of linear logic (MALL), relying on the correctness criterion introduced by Hughes & Van Glabbeek [HG05]. They give a proof of sequentialization in their article, relying on a separation lemma which allows to cut a net in two parts, but possibly deep in the net, meaning we do not cut according to a last rule. This proof is quite elaborated and complex, and no other proof has been given to the best of the authors' knowledge. This is far away from what happens with the unit-free multiplicative fragment of linear logic (MLL), where a plethora of correctness criteria and even more proofs of sequentialization have been given, improving our understanding of MLL proof nets. Henceforth, we propose here another proof of sequentialization for MALL, which is an adaptation of one of the simplest proofs for MLL [Lau13]. This results in a new, bottom-up sequentialization for MALL.

The unit-free multiplicative-additive fragment of linear logic [Gir87] can be defined as follows. Its formulas are given by the following grammar, where X is an atom from a given enumerable set:

$$A,B \coloneqq X \mid X^{\perp} \mid A \otimes B \mid A \parr B \mid A \& B \mid A \oplus B$$

Sequents are sequences of formulas of the form $\vdash A_1, \dots, A_n$ and sequent calculus rules are:[1]

$$\frac{}{\vdash X^{\perp}, X}\,ax \qquad \frac{\vdash \Gamma}{\vdash \sigma(\Gamma)}\,ex(\sigma) \qquad \frac{\vdash A,\Gamma \quad \vdash B,\Delta}{\vdash A \otimes B,\Gamma,\Delta}\,\otimes \qquad \frac{\vdash A,B,\Gamma}{\vdash A \parr B,\Gamma}\,\parr$$

$$\frac{\vdash A,\Gamma \quad \vdash B,\Gamma}{\vdash A \& B,\Gamma}\,\& \qquad \frac{\vdash A,\Gamma}{\vdash A \oplus B,\Gamma}\,\oplus_1 \qquad \frac{\vdash B,\Gamma}{\vdash A \oplus B,\Gamma}\,\oplus_2$$

The main difference with MLL is the &-rule, which introduces the notion of a *slice* [Gir87, Gir96] due to the duplication of the context Γ.

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[1] The axiom rule *ax* is only on atoms; we do not consider the cut rule as we study cut-free MALL; in the exchange rule $ex(\sigma)$, σ is a permutation

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2 Proof nets for unit-free MALL

We use the definition of MALL cut-free proof net of [HG05], as sets of linkings. Other definitions for proof nets exist, see the original one from Girard [Gir96], or others such as [HH16, DG11] for instance. Still, the definition we take is one of the most satisfactory, from the point of view of canonicity and cut elimination for instance (see [HG05, HG16] or the introduction of [HH16] for a comparison of different definitions). We recall here quickly this definition of proof nets. Please refer to [HG05] for more details.

A sequent is seen as its syntactic forest, with as internal vertices its connectives and as leaves the atoms of its formulas. When we write a $⅋\backslash\&$-vertex, we mean a ⅋- or &-vertex (a *negative* vertex); similarly a $\otimes\backslash\oplus$-vertex is a $\otimes$- or $\oplus$-vertex (a *positive* vertex). An *additive resolution* of a MALL sequent Γ is any result of deleting one argument subtree of every additive connectives (& or $\oplus$) of Γ. A *&-resolution* is defined analogously by acting on &-connectives only.

An *(axiom) link* on Γ is a pair of complementary leaves in Γ (labeled with X and $X^\perp$). A *linking* λ on Γ is a set of disjoint links on Γ such that $\cup\lambda$ is the set of leaves of an additive resolution of Γ; this additive resolution is denoted $\Gamma \upharpoonright \lambda$.

A set of linkings Λ on Γ *toggles* a &-vertex W if both premises of W are in $\Gamma \upharpoonright \Lambda = \bigcup_{\lambda\in\Lambda} \Gamma \upharpoonright \lambda$. We say an axiom link *a depends* on a &-vertex W in Λ if there exist $\lambda, \lambda' \in \Lambda$ such that $a \in \lambda \backslash \lambda'$ and W is the only &-vertex toggled by $\{\lambda; \lambda'\}$. The graph $\mathscr{G}_\Lambda$ is defined as $\Gamma \upharpoonright \Lambda$ with the edges from $\cup\Lambda$ and jump edges $l \to W$ for all leaf l and &-vertex W such that there exists $a \in \lambda \in \Lambda$, between l and some l', with a depending on W in Λ. When $\Lambda = \{\lambda\}$ is composed of a single linking, we shall simply denote $\mathscr{G}_\lambda = \mathscr{G}_{\{\lambda\}}$ (which is the graph $\Gamma \upharpoonright \lambda$ with the edges from λ and no jump edges).

A *switch edge* of a $⅋\backslash\&$-vertex N is an in-edge of N, i.e. an edge between N and one of its arguments (a premise) or a jump to N. A *switching cycle* is a cycle with at most one switch edge of each $⅋\backslash\&$.

A *⅋-switching* of λ is any subgraph of $\mathscr{G}_\lambda$ obtained by deleting a switch edge of each ⅋; if we denote by ϕ this choice of edges, the subgraph it yields is $\mathscr{G}_\phi$.

Definition 1. A *MALL cut-free proof net* θ on Γ is a set of linkings satisfying:

(P1) *Resolution*: Exactly one linking of θ is on any given &-resolution of Γ.

(P2) *MLL*: For every ⅋-switching ϕ of every $\lambda \in \theta$, $\mathscr{G}_\phi$ is a tree.

(P3) *Toggling*: Every set $\Lambda \subseteq \theta$ of two or more linkings toggles a & that is in no switching cycle of $\mathscr{G}_\Lambda$.

Condition (P1) is a correctness criterion for ALL proof nets [HG05] and (P2) is the Danos-Regnier criterion for MLL proof nets [DR89]. But putting the two of them is insufficient for the whole MALL, hence the last condition (P3) which takes into account interactions between the slices. A similar condition is also given in some other theories of proof nets with additive connectives, see for instance [DG11]. Sets composed of a single linking λ are not considered in (P3), for by (P2) $\mathscr{G}_\lambda$ has no switching cycle.

An example of proof net, taken from [HG05], is in Figure 1. On the sequent $\Gamma = P_0^\perp \,\&\, P_1^\perp, P_2 \oplus P_3$ (where P_i is an occurrence of the formula P), set $\lambda_1 = \{(P_0^\perp, P_3)\}$ and $\lambda_2 = \{(P_1^\perp, P_2)\}$. One can check that $\theta = \{\lambda_1; \lambda_2\}$ is a MALL cut-free proof net on Γ.

Given a proof tree, building an associated proof net is easy; the reverse direction is where difficulties lie, as we have to recover an ordering from the graph. A MALL cut-free proof net is *sequentializable* if it corresponds to a MALL cut-free proof tree.

Theorem 1 (Sequentialization)**.** *Any MALL cut-free proof net is sequentializable.*

This is proved in [HG05], but the proof is quite complex and relies on finding a separating $⅋\backslash\&$. In this paper, we will adapt a proof of the same theorem from MLL (the fragment of linear logic with $\otimes$ and ⅋ but no & nor $\oplus$) in order to obtain it in another way. We will still use a basic result from [HG05] (proving it from scratch takes half a page).

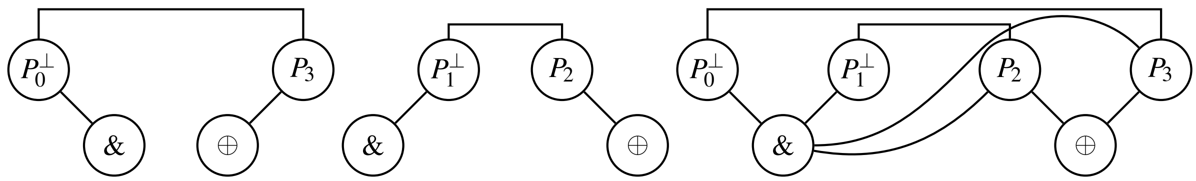

Figure 1: Graphs from a proof net (taken from [HG05]): from left to right $\mathcal{G}_{\lambda_1}$, $\mathcal{G}_{\lambda_2}$ and $\mathcal{G}_\theta$

Lemma 1 (Lemma 4.32 in [HG05])**.** *Every non-empty union S of switching cycles of $\mathcal{G}_\theta$ has a jump out of it: for some leaf $l \in S$ and &-vertex $W \notin S$, there is a jump $l \to W$ in $\mathcal{G}_\theta$.*

3 Method

We follow and extend the method used to prove sequentialization in [Lau13], which is robust enough to apply to MLL with first-order quantifiers as well. We assume given a fixed MALL cut-free proof net θ on an MALL sequent Γ.

Definition 2 (Terminal vertex)**.** A vertex is called *terminal* if it has no conclusion edge in $\mathcal{G}_\theta$; or equivalently if it is associated to a formula A with $\Gamma = \Delta_0, A, \Delta_1$.

There are some vertices that are easy to sequentialize. Our final objective, which is a standard method, is to prove there is always such a vertex, so that we can conclude by induction.

Definition 3 (Sequentializing vertex)**.** A terminal non-leaf vertex V is called *sequentializing* if, depending on its kind:

- $\otimes$-vertex: the removal of V in $\mathcal{G}_\theta$ has two connected components.
- $\oplus$-vertex: the left or right formula tree of V does not belong to $\mathcal{G}_\theta$ (i.e. has no axiom link on any of its leaves in $\mathcal{G}_\theta$).
- $\parr\backslash$&-vertex: a terminal $\parr\backslash$&-vertex is always sequentializing.

Given a path γ, $s(\gamma)$ is its source and $t(\gamma)$ its target. By *simple* path, we mean a path not passing twice through the same vertex, except its source which can be its target, so to allow simple cycles.

Definition 4 (Switching path)**.** A *switching* path is a simple path which does not go (consecutively or not) through two switch edges of a $\parr\backslash$&-vertex.

However, concatenating switching paths may not yield a switching path for two reasons: the concatenation may not be simple, and can take two switch edges of a $\parr\backslash$&-vertex. To obtain a notion stable by concatenation, our prime operation on paths, we consider a restricted notion solving the second problem.

Definition 5 (Strong path)**.** A *strong* path is a switching path which does not start from a $\parr\backslash$&-vertex through one of its switch edges.

Lemma 2. *If γ and γ' are two strong paths, and their concatenation $\gamma\gamma'$ is a simple path (i.e. they are disjoint, non-cyclic and $t(\gamma) = s(\gamma')$), then $\gamma\gamma'$ is a strong path.*

Some dependencies in the proof net are easy to follow: those derived from the sequent forest. We can associate strong paths to them.

Definition 6 (Descending path)**.** For a vertex V, its *descending path* $\delta(V)$ is the unique simple path, in the sequent forest, starting from the conclusion of V and ending with the premise of a terminal vertex. The path $\delta(V)$ is empty if and only if V is terminal.

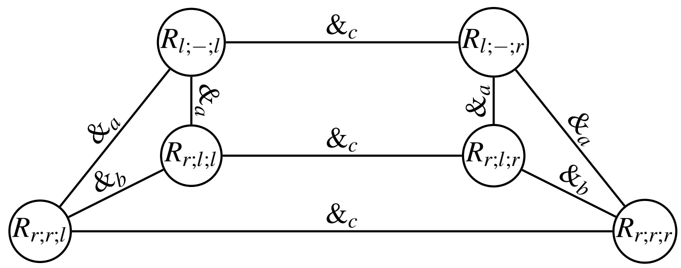

Figure 2: &-graph $\mathscr{W}_\Gamma$ of $\Gamma = A \&_a (A \&_b A), A^\perp \&_c A^\perp$
$R_{l;-;r}$ means we chose the left premise (l) of $\&_a$, $\&_b$ is absent ($-$) and the right premise (r) of $\&_c$

The content so far is mostly coming from [HG05], and our contribution is proving Theorem 1 by the following method:

1. for a terminal non-sequentializing vertex, find a negative vertex which prevents it to be sequentializing, called a correctness $⅋\backslash\&$.
2. use these vertices to build a particular strong path, called *critical*, until reaching a sequentializing vertex or a switching cycle.
3. continuing the critical path when arriving on a switching cycle, thus having a path ending on a sequentializing vertex.
4. proving sequentialization thanks to sequentializing vertices.

Step (4) is standard and thus omitted in this paper. Steps (1) & (2) follow the proof on MLL. Step (3) is where complexity from MALL comes into play, for there are no switching cycles in an MLL proof net.

4 &-graph $\mathscr{W}_\Gamma$ of a sequent Γ

We give here a graphical representation of &-resolutions allowing easier visualization of notions like toggling and dependency.

Definition 7 ($\mathscr{W}_\Gamma$)**.** Given Γ a MALL sequent, $\mathscr{W}_\Gamma$ is a simple undirected graph defined as follows:

- $\mathscr{W}_\Gamma$ has for vertices the &-resolutions of Γ.
- for R and S two &-resolutions of Γ, there is an edge R—S in $\mathscr{W}_\Gamma$ if there is exactly one &-vertex W with different choices of premise in R and S; we label this edge with this vertex, yielding $R \overset{W}{—} S$.

For instance, the graph $\mathscr{W}_\Gamma$ with $\Gamma = A \&_a (A \&_b A), A^\perp \&_c A^\perp$ is depicted on Figure 2. This graph represents the increase in complexity between MALL and MLL, because an MLL proof net is a single point on it. Useful paths to consider in this graph are those following togglings.

Definition 8. A *toggling path* in $\mathscr{W}_\Gamma$ is a path between &-resolutions R and S such that the labels of its edges are exactly the &-vertices present in both R and S with different choices of premises, each taken exactly once.

Lemma 3. *Given two &-resolutions, there exists a toggling path between them in* $\mathscr{W}_\Gamma$*.*

The graph $\mathscr{W}_\Gamma$ has other interesting properties, e.g. given an orientation $o \in \{\text{left}; \text{right}; \text{absent}\}$ and a &-vertex W, the subgraph of $\mathscr{W}_\Gamma$ induced by the set of &-resolutions where W has orientation o is connected with respect to toggling paths. Another is that the set of all &-resolutions on which a given linking λ is on induces a connected subgraph of $\mathscr{W}_\Gamma$.

Given a MALL proof net θ on Γ and a &-resolution R, we set $\lambda(R)$ the unique linking of θ on R according to (P1). Toggling paths are quite useful to find jump edges, i.e. links depending on &s.

Lemma 4. *Let P be a terminal non-sequentializing $\oplus$-vertex of a MALL cut-free proof net θ. Then there exist axiom links a and b, using respectively a left-ancestor and a right-ancestor of P in the formula tree, a &-vertex W and $\lambda_a, \lambda_b \in \theta$ such that $a \in \lambda_a \backslash \lambda_b$, $b \in \lambda_b \backslash \lambda_a$ and W is the only & toggled by $\{\lambda_a; \lambda_b\}$.*

Proof. As P is non-sequentializing, there exist linkings $\lambda_{a'}, \lambda_{b'} \in \theta$ and axiom links $a' \in \lambda_{a'}$ and $b' \in \lambda_{b'}$ using respectively a leaf in the left and right formula tree of P. We take $R_{a'}$ and $R_{b'}$ resolutions on which $\lambda_{a'}$ and $\lambda_{b'}$ are respectively, and μ a toggling path between them (Lemma 3). For P is terminal, any linking has an axiom link using a leaf in the formula tree of P, either left or right (according to its additive resolution). As $\lambda_{a'}$ on $R_{a'}$ is in the left case and $\lambda_{b'}$ on $R_{b'}$ in the right one, we must have an edge $R_a \overset{W}{—} R_b$ in μ such that $\lambda(R_a)$ has an axiom link a with a leaf in the left formula tree of P, and $\lambda(R_b)$ a link b with a leaf in the right one. As a and b belong to different additive resolutions, we have more precisely $a \in \lambda(R_a) \backslash \lambda(R_b)$ and $b \in \lambda(R_b) \backslash \lambda(R_a)$. But W is the only & with different choices of premise in R_a and R_b, and $\lambda(R_a) \neq \lambda(R_b)$. Thus W is the only & toggled by $\{\lambda(R_a); \lambda(R_b)\}$. □

5 Sketch of the proof

In all that follows, θ is a MALL cut-free proof net. We now define the main tool to find sequentializing vertices from non-sequentializing ones.

Definition 9 (Correctness ⅋\&)**.** Let P be some terminal $\otimes\backslash\oplus$-vertex. A *correctness* ⅋\& for P is a ⅋\&-vertex N with two disjoint strong paths κ and κ' from P to N (called *correctness paths*) which both start with a premise of P and end with a switch edge of N.

A triple (κ, κ', N) is called a *correctness triple* for P and a pair (κ, N) a *correctness pair*.

Intuitively, N is a correctness ⅋\& for P means that N must be done before P in some slices of the proof. The correctness paths indicate this dependency. The first step in our proof is to prove that terminal non-sequentializing vertices have such correctness ⅋\&.

Lemma 5. *Non-sequentializing terminal $\oplus$-vertices have correctness* &.

Lemma 6. *Non-sequentializing terminal $\otimes$-vertices have correctness ⅋\& or are in a switching cycle.*

The proof of Lemma 5 relies on finding jump edges with Lemma 4. The proof of Lemma 6 is more technical, especially when the $\otimes$-vertex T we consider depends on a &: in this case, we have to find a slice λ where this dependency matters, as just considering other slices T may be sequentializing.

We then use correctness triples to build a strong path called critical, illustrated on Figure 3.

Definition 10 (Critical path)**.** A *critical path* γ is a strong path which can be decomposed into the concatenation $\delta_0\kappa_0\delta_1\kappa_1 \ldots \delta_{n-1}\kappa_{n-1}\delta_n$ (for some $n \in \mathbb{N}$) while respecting the following conditions, denoting $N_i = s(\delta_i) = t(\kappa_{i-1})$ and $P_i = t(\delta_i) = s(\kappa_i)$:

- (κ_i, N_{i+1}) is a correctness pair for the terminal non-sequentializing vertex P_i.
- $\delta_i = \delta(N_i)$ for $0 \leq i < n$.
- $\delta_n \subseteq \delta(N_n)$ and no vertex of δ_n is in a switching cycle, except maybe its target P_n (δ_n may be the empty path, in which case $P_n = N_n$).

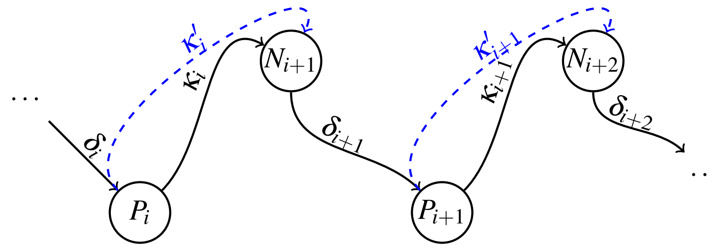

Figure 3: Critical path, with its auxiliary correctness paths dashed

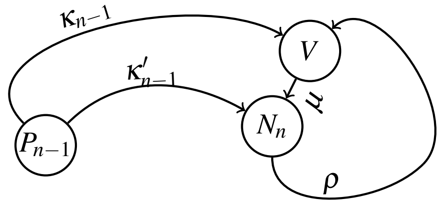

Figure 4: Only case where adding a strong path ρ to a critical path creates a cycle

- no N_i nor P_i is in a switching cycle, except maybe P_n.

Given a critical path γ, we can fix such a decomposition and a family (κ'_i) of paths, called *auxiliary correctness paths of* γ, such that each $(\kappa_i, \kappa'_i, N_{i+1})$ is a correctness triple for P_i.

The reason we insist on the absence of switching cycles is that they represent dependencies across slices. A switching cycle containing vertices A and B is intuited as a representation that, in the proof, A must come before B in some slices, and B before A in some others [HG05]. As we can have P and one of its correctness ⅋\& N in a switching cycle, we need to ensure this is not the case, so that the dependencies we follow hold no matter the slice. This corresponds in $\mathscr{G}_\theta$ to a condition ensuring we cannot go back on our critical path, which allows to concatenate a path to it. This condition is given by the following technical lemma, which basically says that only a particular form of path, given in Figure 4, cannot be concatenated to a critical path, and will instead create a cycle.

Lemma 7. *Let γ be a critical path in $\mathscr{G}_\theta$ (with notations of Definition 10). If there is a strong path ρ' from $t(\gamma)$ to $\gamma \backslash \{t(\gamma)\}$, disjoint from γ, then γ stops with κ_{n-1} (whence $t(\gamma) = N_n$ and δ_n is empty), and there is a non-empty non-cyclic strong prefix ρ of ρ', disjoint from all δ_i, κ_i and κ'_i, from N_n to $\kappa_{n-1} \backslash \{P_{n-1}\}$ (or $\kappa'_{n-1} \backslash \{P_{n-1}\}$). If we denote $V = t(\rho)$ and μ the subpath of κ_{n-1} (or κ'_{n-1}) from V to N_n, then $\rho\mu$ is a switching cycle containing N_n. Moreover, V is a ⅋\&-vertex to which ρ and κ_{n-1} (or κ'_{n-1}) arrive by two of its switch edges.*

Definition 11 (Maximal critical path)**.** A critical path γ is said to be *maximal*[2] if its target $t(\gamma)$ is a sequentializing vertex or if $t(\gamma)$ belongs to a switching cycle.

We can build a maximal critical path from any chosen non-leaf vertex, simply by using correctness triples given by Lemmas 5 and 6 and concatenating them thanks to Lemma 7, until reaching a sequentializing vertex or a switching cycle. As we stop once reaching a switching cycle, the case on Figure 4 cannot happen during the construction.

[2]This corresponds to maximality when comparing critical paths with the same source with respect to the prefix ordering.

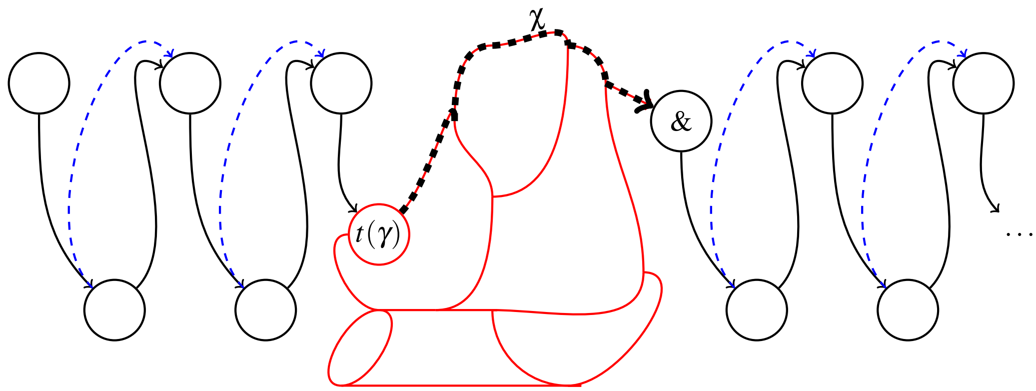

Figure 5: Strong path built in the proof of Lemma 9, with critical paths in black, their auxiliary correctness paths dashed in blue and connected unions of switching cycles in red

Thus, starting from an arbitrary non-leaf vertex, we reach a sequentializing vertex or a switching cycle. In the former case, we are done. In the latter, we reach a zone of dependency where we have to take into account that the order of sequentialization may depend on slices. To go out of it, we have to find a & which causes this dependency, and do it recursively as this & may cause it only in certain slices.

We will consider unions of switching cycles, as the correctness criterion (P3) allows us to go out of them through Lemma 1, by finding which & is causing these cycles. However, to extend our path into a strong path, we have to ask more for these unions: their connectedness with respect to strong paths.

Definition 12. A subgraph S is said to be *strongly connected* if for all vertices $u \neq v$ in S, there exists inside S a strong path from u to v.

Lemma 8. *A union of switching cycles is connected if and only if it is strongly connected.*

Next, we can extend a critical path through a connected union of switching cycles, while forbidding to go back on already visited vertices. Intuitively, a critical path follows dependencies holding in all slices, until reaching a switching cycle where dependencies change between slices. Connected union of switching cycles will then be exhibited and grown up, until we can go out by finding a & which created these cycles (see Figure 5).

Lemma 9. *Let γ be a maximal critical path in $\mathscr{G}_\theta$. If γ does not end on a sequentializing vertex, then there exists a* non-empty *non-cyclic strong path χ such that:*

- *$\xi = \gamma\chi$ is a non-cyclic strong path.*
- *$t(\xi)$ is a &-vertex to which ξ arrives through one of its switch edges.*
- *there is no strong path α from $t(\xi)$ to $\xi \setminus \{t(\xi)\}$, with α disjoint from ξ.*

The last condition ensures we do not go back on our path. The main difficulty in this proof is that we have two cases when reaching a switching cycle: either this cycle contains only the target of the critical path, which is the easy case; or we are in the situation depicted on Figure 4, and have to use the fact that our cycles can include the end of our path, but no vertices before P_{n-1}. In both cases, noting S the connected union of switching cycles we consider, we use Lemma 1 to find a leaf $l \in S$ and a &-vertex $W \notin S$ such that the jump $l \to W$ is in $\mathscr{G}_\theta$. Connectedness allows us to find a strong path μ in S from the target of γ to l, then a strong path $\chi = \mu(l \to W)$ to W, as illustrated on Figure 5. If there is a strong path from W to S, we make our union of switching cycles S bigger by adding W and this last path. Otherwise the wished conditions hold and we can conclude.

We then prove there is a strong path ending on a sequentializing vertex. The condition on non-emptiness in Lemma 9 ensures we progress during the construction. Concluding by proving Theorem 1 thanks to sequentializing vertices is well-established routine [Gir87].

6 Conclusion

This new proof of sequentialization for MALL is adapted from the one of MLL: we found correctness triples and had to take into account the presence of switching cycles, but otherwise the proof stays the same. We can extend it straightforwardly in the presence of cut, in a similar fashion as in [HG05]. Another possible step would be extending the method to MALL with mix.

A noteworthy use of this new proof is to look for new correctness criteria, so as to better understand interactions between additives and multiplicatives, with the hope of better understanding interactions between additives and quantifiers, or additives and exponentials, and then develop canonical proof nets outside the fragments they are confined in. In this proof of sequentialization we use Lemma 1, and so (P3), only on connected unions of switching cycles, which yields a seemingly weaker but equivalent version of (P3): "For any set Λ of two or more linkings of θ and any non-empty connected union of switching cycles S of $\mathscr{G}_\Lambda$, Λ toggles a & that is not in S.". This equivalence can be proved directly, using Lemma 8. This weaker version was not apparent in the proof of [HG05], where Lemma 1 is used on arbitrary unions. However, this does not answer their *Single Cycle Conjecture*, which states (P3) is equivalent to "For any set Λ of two or more linkings of θ and any switching cycle S of $\mathscr{G}_\Lambda$, Λ toggles a & that is not in S.", conjecture which still needs to be addressed.

References

[DG11] Paolo Di Giamberardino (2011): *Jump from Parallel to Sequential Proofs: Additives*. Technical Report. Available at https://hal.archives-ouvertes.fr/hal-00616386.

[DR89] Vincent Danos & Laurent Regnier (1989): *The structure of multiplicatives*. *Archive for Mathematical Logic* 28, pp. 181–203, doi:10.1007/BF01622878.

[Gir87] Jean-Yves Girard (1987): *Linear logic*. *Theoretical Computer Science* 50, pp. 1–102, doi:10.1016/0304-3975(87)90045-4.

[Gir96] Jean-Yves Girard (1996): *Proof-nets: the parallel syntax for proof-theory*. In Aldo Ursini & Paolo Agliano, editors: *Logic and Algebra*, *Lecture Notes In Pure and Applied Mathematics* 180, Marcel Dekker, New York, pp. 97–124, doi:10.1201/9780203748671-4.

[HG05] Dominic Hughes & Rob van Glabbeek (2005): *Proof Nets for Unit-free Multiplicative-Additive Linear Logic*. *ACM Transactions on Computational Logic* 6(4), pp. 784–842, doi:10.1145/1094622.1094629.

[HG16] Dominic Hughes & Rob van Glabbeek (2016): *MALL proof nets identify proofs modulo rule commutation*. *CoRR*. arXiv:1609.04693.

[HH16] Dominic Hughes & Willem Heijltjes (2016): *Conflict nets: Efficient locally canonical MALL proof nets*. In M. Grohe, E. Koskinen & N. Shankar, editors: *Proceedings of the 31st ACM/IEEE Symposium on Logic in Computer Science*, ACM, New York NY USA, pp. 437–446, doi:10.1145/2933575.2934559.

[Lau13] Olivier Laurent (2013): *Sequentialization of Multiplicative Proof Nets*. Available at https://perso.ens-lyon.fr/olivier.laurent/seqmll.pdf. Unpublished note.