Nonlinearity-Aware Partial-Update Schmidt Kalman Filter

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***Abstract*—The partial-update filter is a Kalman filter modification that can accommodate higher nonlinearities and uncertainties than a nominal and Schmidt-Kalman filter. This robustness enhancement of the partial-update filter is attributed to its capability to limit the impact of incorrect updates by applying user-selected static percentages of the nominal Kalman update, to user-selected states at any time step.**

To further extend the partial-update capabilities and applicability, this paper presents two methods for dynamically and automatically selecting the partial-update percentages based on nonlinearity metrics of the process and measurement model. By enabling dynamic update percentages, the filter automatically leverages situations where higher updates can be applied and lower updates are deemed suitable. This leads to higher statistical consistency and accuracy with respect to the nominal Kalman and static partial-update filters. The superior accuracy and consistency of the proposed nonlinearity-aware partial-update methods are shown via a numerical example.

***Index Terms*—Kalman filter, Schmidt, Robust, Partial-update, Nonlinearity**

I. Introduction

The extended Kalman filter (EKF) is one of the most used filtering algorithms for nonlinear state estimation mainly because it is functional for many systems, straightforward to implement, and runs online [1]. However, when some state vector elements are significantly less observable than others, and the system is highly nonlinear, a conventional EKF can produce estimate degradation, filter inconsistency, and even divergence [2] [3]. An immediate, straightforward, and often effective alternative to ameliorate difficulties caused by nonlinearities and observability disparities, is to modify the Kalman filter to turn it into a Schmidt filter [2], [4]–[6]. In practice, the Schmidt filter is identical to the Kalman filter, except the Schmidt filter only updates a user-selected part of the state and the remaining state elements and covariances are unchanged. In the literature, the Schmidt filter partitions the state into core and considered (not updated) states. Core states are often those of primary interest, like position, velocity, and attitude. The considered states are often identified as those that, estimated or not, degrade the conventional Kalman filter estimates; they are typically constant parameters, slow-varying biases, and/or states with relatively low observability [2]. Although the Schmidt filter does not update the considered states, the states and associated uncertainties are still considered in the filtering solution [7].

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Even when the Schmidt modification can benefit a Kalman filter, it constrains it to systems where the considered (non-estimated) states are constant over the life of the filter run. A more general Schmidt filter is the partial-update Schmidt filter [8]. The partial-update technique generalizes the Schmidt filter, allowing the application of either a nominal (100%), zero (0%), or a partial-update to any state at any time step. In this way, the partial-update filter can cope with varying parameters, drifting biases, and non-constant poorly observable states [8]–[11].

To use the partial-update filter, one must establish the states to be partially updated and the updating percentages for each state. To date, the percentage update can be selected manually as in [8], [12]–[17], based on the observed simulated or experimental system trajectories, or automatically selected, based on the local observability of each partially-updated state [18]. Although the observability-based method is a method to appropriate the partial-update percentages online by leveraging its awareness of the system's observability fluctuation, it still depends on the accuracy of the system linearization. In other words, it entirely relies on the assumption that the magnitude of high-order effects is minor, and thus the observability metrics being used remain within a valid regime. To directly assess the effect of nonlinearities and complement the observability-based method from [18], in this paper, we propose two approaches that directly monitor high-order terms and use this information to establish the partial-update percentages: a nonlinearity-aware method and a nonlinear covariance-aware method. Both nonlinearity-based methods aim to decrease the partial update when high-order terms are significant and increase it if high-order terms are not comparable to first-order terms. Overall, the nonlinearity-aware and nonlinear covariance-aware methods intend to leverage as much update information as possible, and to reduce the negative impact of incorrect updates when the state is in locally highly-nonlinear regions.

The remainder of this paper is organized as follows. Section II presents key background information and establishes relevant notation. The specifics of the nonlinearity-aware methods for dynamic selection of the partial-update percentages are discussed in Section III. Numerical simulations of both methods within an extended Kalman filter are presented in Section IV. Finally, Section V presents the conclusions of this work.

II. Background

A. Kalman filter notation

This section establishes the EKF nomenclature and notation relevant to the partial-update developments presented in this paper. This notation is the same used in [19]. Consider the following discrete nonlinear dynamic system with state vector $\mathbf{x}_k \in \mathbb{R}^n$, known input $\mathbf{u}_k \in \mathbb{R}^r$, measurement vector $\tilde{\mathbf{y}}_k \in \mathbb{R}^m$, process noise $\mathbf{w}_k \in \mathbb{R}^q$, and measurement noise $\mathbf{v}_k \in \mathbb{R}^m$:

$$\mathbf{x}_k = \mathbf{f}_{k-1}(\mathbf{x}_{k-1}, \mathbf{u}_{k-1}) + \mathbf{G}_{k-1}\mathbf{w}_{k-1} \tag{1}$$

$$\tilde{\mathbf{y}}_k = \mathbf{h}_k(\mathbf{x}_k) + \mathbf{v}_k \tag{2}$$

$$\mathbf{w}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}_k) \tag{3}$$

$$\mathbf{v}_k \sim \mathcal{N}(\mathbf{0}, \mathbf{R}_k) \tag{4}$$

Here, $\mathbf{w}_k$ and $\mathbf{v}_k$ are zero-mean Gaussian white-noise processes, with covariances $\mathbf{Q}_k = \mathbf{E}[\mathbf{w}_k\mathbf{w}_k^{\mathrm{T}}]$ and $\mathbf{R}_k = \mathbf{E}[\mathbf{v}_k\mathbf{v}_k^{\mathrm{T}}]$, respectively; the function $\mathbf{h}(\mathbf{x}_k) = \begin{bmatrix} h_1(\mathbf{x}_k) & h_2(\mathbf{x}_k) & \dots & h_m(\mathbf{x}_k)\end{bmatrix}^{\mathrm{T}}$, is the measurement model function that maps $\mathbb{R}^n \to \mathbb{R}^m$, and all of the sub-indices denote the time step. Also, consider the EKF discrete covariance propagation equation given as

$$\mathbf{P}_k^- = \mathbf{F}_{k-1}\mathbf{P}_{k-1}^+\mathbf{F}_{k-1}^{\mathrm{T}} + \mathbf{G}_{k-1}\mathbf{Q}_{k-1}\mathbf{G}_{k-1}^{\mathrm{T}} \tag{5}$$

Where $\mathbf{P}_k^-$ is the prior error covariance, $\mathbf{P}_{k-1}^+$ is the most recent updated covariance, $\mathbf{G}_{k-1} = \frac{\partial \mathbf{f}_{k-1}}{\partial \mathbf{w}_k}\Big|_{\mathrm{E}[\mathbf{w}_{k-1}]}$ is the $n \times q$ matrix mapping the process noise from the vector $\mathbf{w}_k$ to the state, and $\mathbf{F}_{k-1} = \frac{\partial \mathbf{f}_{k-1}}{\partial \mathbf{x}}\Big|_{\mathrm{E}[\mathbf{x}_{k-1}]}$ is the process model Jacobian.

When an observation is available, the measurement update step is performed through the Kalman gain $\mathbf{K}_k$ according to Equations (6)-(9). Note that the measurement Jacobian at time k, is denoted by $\mathbf{H}_k$ [19]:

$$\mathbf{K}_k = \mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}}(\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k)^{-1} \tag{6}$$

$$\mathbf{P}_k^+ = (\mathbf{I} - \mathbf{K}_\mathbf{k}\mathbf{H}_k)\mathbf{P}_k^- \tag{7}$$

$$\hat{\mathbf{y}}_k = \mathbf{h}(\hat{\mathbf{x}}_k^-, k) \tag{8}$$

$$\hat{\mathbf{x}}_k^+ = \hat{\mathbf{x}}_k^- + \mathbf{K}_k(\tilde{\mathbf{y}}_k - \hat{\mathbf{y}}_k) \tag{9}$$

where

$$\mathbf{H}_k = \frac{\partial \mathbf{h}_k}{\partial \mathbf{x}}\Big|_{\hat{\mathbf{x}}_k^-} \tag{10}$$

Here the hat notation, i.e., $[\hat{\cdot}]$, denotes an expected or estimated value. The notations $[\,\cdot\,]^+$ and $[\,\cdot\,]^-$ refer to posterior and prior values, respectively. The set of equations (1) to (10), constitute the extended Kalman filter framework where the partial-update concepts are applied in this paper.

B. The partial-update implementation

The implementation of the partial-update concept within the EKF is straightforward. It consists of a weighted sum of the prior state and covariance estimates ($\hat{\mathbf{x}}^-$, $\mathbf{P}^-$), with their corresponding posteriors ($\hat{\mathbf{x}}^+$, $\mathbf{P}^+$); the element-wise expression of the partial-update for the state and covariance estimates, at a given measurement step, is given by [8]

$$\hat{\mathbf{x}}_i^{++} = \gamma_i\hat{\mathbf{x}}_i^- + (1-\gamma_i)\hat{\mathbf{x}}_i^+ \tag{11}$$

$$\mathbf{P}_{ij}^{++} = \gamma_i\gamma_j\mathbf{P}_{ij}^- + (1-\gamma_i\gamma_j)\mathbf{P}_{ij}^+ \tag{12}$$

$$\gamma_i = 1 - \beta_i \tag{13}$$

or in matrix form as

$$\mathbf{x}^{++} = \mathbf{\Gamma}\hat{\mathbf{x}}^- + (\mathbf{I} - \mathbf{\Gamma})\hat{\mathbf{x}}^+ \tag{14}$$

$$\mathbf{P}^{++} = \mathbf{\Gamma}(\mathbf{P}^- - \mathbf{P}^+)\mathbf{\Gamma} + \mathbf{P}^+ \tag{15}$$

Here, the scalars (weights or percentages) $\beta_i \in [0,1]$ in Equation (13), represent the percentage of the nominal update to be applied. The product $\gamma_i\gamma_j$, and β_i, are defined for $i,j = 1,\dots,n$, where n is the total number of states in the filter. If $\beta_i = 0$, the i^{th} state is not updated; this is equivalent to using a Schmidt (or consider) filter on the i^{th} state. On the other hand, if $\beta_i = 1$, the i^{th} state is updated via a regular Kalman filter (full) update. Setting β_i anywhere in between applies a partial update. The notation $[\,\cdot\,]^{++}$ in the partial-update equations identifies the state and covariance that will be used at the next propagation step after a measurement update has been performed. Last, in the matrix Equation (15), $\mathbf{\Gamma} = \text{diag}\begin{bmatrix}\gamma_1 & \dots & \gamma_n\end{bmatrix}$.

Note that the partial-update expressions may be written differently, but the one used here facilitates discussion and mathematical manipulations. Also, note that a partial-update can be applied to any state. This means that even main or core states (e.g. attitude, position, velocity) can receive an update percentage of less than 100 %. In fact, some systems that use the partial-update concept have shown that it can be beneficial that core states receive update percentages slightly lower than 100% as it helps to limit the adverse effects that nonlinearities can exacerbate during an update. Examples of such systems can be found in [15], [20], and [9].

III. Dynamic partial-update methods

For many low-uncertainty applications, the tolerance of the EKF to slight mismodelling is sufficient to prevent divergence. However, suppose the higher-order effects of a system are significant. In that case, the first-order EKF update becomes less optimal, and the filter estimates can be quickly degraded due to the more considerable mismatch between the system and the filter's model. Based on this fact, limiting the EKF update when the high-order effects are comparable to first-order effects seems reasonable. In this spirit, this section proposes two ways to appropriate the partial-update percentages or β weights dynamically: the nonlinearity-aware (DNL) and the covariance nonlinearity-aware (DC) methods.

A. Nonlinearity-aware based method

The nonlinearity-aware method monitors the Kalman second-order to first-order terms ratio to determine the partial-update percentages. The idea of this approach is to limit the EKF update as this ratio increases. To establish the relationship between second and first-order terms and the partial-update weights, consider the equations for the discrete second-order Kalman filter (EKF2) [3]. For the dynamics and uncertainty propagation, with EKF2's variables identified with the sub-index (2), we have

$$\hat{\mathbf{x}}^-_{k(2)} = f(\hat{\mathbf{x}}^-_{k-1}, \mathbf{u}_{k-1}, k-1) + \frac{1}{2}\sum_{i=1}^{n} \phi_i \mathbf{tr}\left[\frac{\partial^2 f_i}{\partial x^2}\Big|_{\hat{\mathbf{x}}^+_{k-1}} \mathbf{P}^+_{k-1} \right] \tag{16}$$

$$\mathbf{P}^-_{k(2)} = \mathbf{F}_{k-1}\mathbf{P}^+_{k-1}\mathbf{F}^{\mathrm{T}}_{k-1} + \mathbf{G}_{k-1}\mathbf{Q}_{k-1}\mathbf{G}^{\mathrm{T}}_{k-1} \tag{17}$$

Whereas the measurement update equations are given by

$$\hat{\mathbf{x}}^+_{k(2)} = \hat{\mathbf{x}}^-_{k(2)} + \mathbf{K}_{k(2)}\left[\tilde{\mathbf{y}}_k - \mathbf{h}(\hat{\mathbf{x}}^-_{k(2)}\ ,\ k) \right] - \pi \tag{18}$$

$$\pi = \frac{1}{2}\mathbf{K}_{k(2)}\sum_{i=1}^{m} \phi_i \mathbf{tr}\left[\mathbf{D}_{k,i}\mathbf{P}^-_{k(2)} \right] \tag{19}$$

$$\mathbf{D}_{k,i} = \frac{\partial^2 h_i(\mathbf{x}_k, k)}{\partial x^2}\Big|_{\hat{\mathbf{x}}^-_{k(2)}} \tag{20}$$

$$\mathbf{K}_{k(2)} = \mathbf{P}^-_{k(2)}\mathbf{H}^{\mathrm{T}}_k(\mathbf{H}_k\mathbf{P}^-_{k(2)}\mathbf{H}^{\mathrm{T}}_k + \mathbf{R}_k)^{-1} \tag{21}$$

$$\mathbf{P}^+_{k(2)} = (\mathbf{I} - \mathbf{K}_{k(2)}\mathbf{H}_k)\mathbf{P}^-_{k(2)} \tag{22}$$

Here, $\mathbf{D}_{k,i}$ is the Hessian matrix for the i^{th} measurement element of $\mathbf{h}$ at time k, $\mathbf{F}$ and $\mathbf{H}$, the process and measurement model Jacobian, are defined as before, and ϕ_i is the single-entry vector (with a 1 at the i^{th} element) given by,

$$\phi_i^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & \dots & 0 & \dots & 1 & \dots & 0 \end{bmatrix}^{\mathrm{T}} \tag{23}$$

Next, we establish a relationship between the second-order EKF2 terms' influence and the partial-update percentages. To do so, consider the measurement update from Equation (18) and the vector $\boldsymbol{\pi}$ expressed together as

$$\hat{\mathbf{x}}^+_{k(2)} = \hat{\mathbf{x}}^-_{k(2)} + \mathbf{K}_{k(2)}\left[\tilde{\mathbf{y}}_k - \mathbf{h}(\hat{\mathbf{x}}^-_{k(2)}\ ,\ k) \right] - \frac{1}{2}\mathbf{K}_{k(2)}\sum_{i=1}^{m} \phi_i \mathbf{tr}\left[\mathbf{D}_{k,i}\mathbf{P}^-_{k(2)} \right] \tag{24}$$

Further, let the prior state, $\hat{\mathbf{x}}^-_k$, as defined in Equation (16) be substituted into the previous equation to form

$$\hat{\mathbf{x}}^+_{k(2)} = f(\hat{\mathbf{x}}^-_{k-1}, \mathbf{u}_{k-1}, k-1) + \frac{1}{2}\sum_{i=1}^{n} \phi_i \mathbf{tr}\left[\frac{\partial^2 f_i}{\partial x^2}\Big|_{\hat{\mathbf{x}}^+_{k-1}} \mathbf{P}^+_{k-1} \right] + \mathbf{K}_{k(2)}\left[\tilde{\mathbf{y}}_k - \mathbf{h}(\hat{\mathbf{x}}^-_{k(2)}\ ,\ k) \right] - \frac{1}{2}\mathbf{K}_{k(2)}\sum_{i=1}^{m} \phi_i \mathbf{tr}\left[\mathbf{D}_{k,i}\mathbf{P}^-_{k(2)} \right] \tag{25}$$

By reorganizing the terms the posterior estimate, $\hat{\mathbf{x}}^+_{k(2)}$, can be written as,

$$\hat{\mathbf{x}}^+_{k(2)} = f(\hat{\mathbf{x}}^-_{k-1}, \mathbf{u}_{k-1}, k-1) + \mathbf{K}_{k(2)}\left[\tilde{\mathbf{y}}_k - \mathbf{h}(\hat{\mathbf{x}}^-_{k(2)}\ ,\ k) \right] + \mathbf{Y} \tag{26}$$

where

$$\mathbf{Y} = \frac{1}{2}\left\{ \sum_{i=1}^{n} \phi_i \mathbf{tr}\left[\frac{\partial^2 f_i}{\partial x^2}\Big|_{\hat{\mathbf{x}}^+_{k-1}} \mathbf{P}^+_{k-1} \right] - \mathbf{K}_{k(2)}\sum_{i=1}^{m} \phi_i \mathbf{tr}\left[\mathbf{D}_{k,i}\mathbf{P}^-_{k(2)} \right] \right\} \tag{27}$$

Recalling the partial-update matrix expression for the state and using Equation (9),

$$\mathbf{x}^{++}_k = \hat{\mathbf{x}}^-_k + (\mathbf{I} - \boldsymbol{\Gamma})\mathbf{K}_k(\tilde{\mathbf{y}}_k - \hat{\mathbf{y}}_k) \tag{28}$$

and expressing it in terms of the function dynamics, measurement function (for the same assumed system in the EKF2), and expanding it, leads to

$$\begin{aligned} \mathbf{x}^{++}_k &= \hat{\mathbf{x}}^-_k + (\mathbf{I} - \boldsymbol{\Gamma})\mathbf{K}_k(\tilde{\mathbf{y}}_k - \hat{\mathbf{y}}_k) \\ &= \hat{\mathbf{x}}^-_k + \mathbf{K}_k(\tilde{\mathbf{y}}_k - \hat{\mathbf{y}}_k) - \boldsymbol{\Gamma}\mathbf{K}_k(\tilde{\mathbf{y}}_k - \hat{\mathbf{y}}_k) \\ &= f(\hat{\mathbf{x}}^-_{k-1}, \mathbf{u}_{k-1}, k-1) + \mathbf{K}_k(\tilde{\mathbf{y}}_k - \hat{\mathbf{y}}_k) - \boldsymbol{\Gamma}\mathbf{K}_k(\tilde{\mathbf{y}}_k - \hat{\mathbf{y}}_k) \\ &= f(\hat{\mathbf{x}}^-_{k-1}, \mathbf{u}_{k-1}, k-1) + \mathbf{K}_k\left[\tilde{\mathbf{y}}_k - \mathbf{h}(\hat{\mathbf{x}}^-_k\ \ k) \right] - \\ &\qquad \boldsymbol{\Gamma}\mathbf{K}_k\left[\tilde{\mathbf{y}}_k - \mathbf{h}(\hat{\mathbf{x}}^-_k\ ,\ k) \right] \end{aligned} \tag{29}$$

Next, it is assumed that the partial-update filter compensates the state and covariance estimates sufficiently to maintain them close to the second-order state and covariance estimates. That is $\mathbf{x}_{k(2)} \approx \mathbf{x}_k$, $\mathbf{K}_{k(2)} \approx \mathbf{K}_k$, $\mathbf{P}_{k(2)} \approx \mathbf{P}_k$. Considering this, a direct term-by-term comparison of the partial-update expression from Equation (29) and Equation (26), reveals that the term with the partial-update weights,

$$-\boldsymbol{\Gamma}\mathbf{Z} := -\boldsymbol{\Gamma}\mathbf{K}_k\left[\tilde{\mathbf{y}}_k - \mathbf{h}(\hat{\mathbf{x}}^-_k\ ,\ k) \right] \tag{30}$$

can be directly related to second-order terms of the EKF2 as follows:

$$-\boldsymbol{\Gamma}\mathbf{K}_k\left[\tilde{\mathbf{y}}_k - \mathbf{h}(\hat{\mathbf{x}}^-_k\ ,\ k) \right] \propto \frac{1}{2}\left\{ \sum_{i=1}^{n} \phi_i \mathbf{tr}\left[\frac{\partial^2 f_i}{\partial x^2}\Big|_{\hat{\mathbf{x}}^+_{k-1}} \mathbf{P}^+_{k-1} \right] - \mathbf{K}_k\sum_{i=1}^{m} \phi_i \mathbf{tr}\left[\mathbf{D}_{k,i}\mathbf{P}^-_k \right] \right\} \tag{31}$$

or

$$-\boldsymbol{\Gamma}\mathbf{K}_k\left[\tilde{\mathbf{y}}_k - \mathbf{h}(\hat{\mathbf{x}}^-_k\ ,\ k) \right] \propto \mathbf{Y} \tag{32}$$

Note that in relation (32), the sub-index (2) is dropped due to the assumption that the partial-update produces estimates close to those of the EKF2. Also note that this expression follows the previously discussed idea of selecting $\gamma_i \in [0, 1]$, according to the second-order terms influence since it suggests that:

- $\boldsymbol{\Gamma}$ should be set with high values if the second-order effects, $\mathbf{Y}$, are large.
- If $\mathbf{Y}$ is small, $\boldsymbol{\Gamma}$ is to be set with small values.

Noticing that both left and right terms of expression (31) are $n \times 1$ vectors, individual relationships between second-order terms and the j^{th} partial-update weights can be established as,

$$\boldsymbol{\Gamma}_{jj} \propto \frac{\frac{1}{2}\left\{ \sum_{i=1}^{n} \phi_i \mathbf{tr}\left[\frac{\partial^2 f_i}{\partial x^2}\Big|_{\hat{\mathbf{x}}^+_{k-1}} \mathbf{P}^+_{k-1} \right] - \mathbf{K}_k\sum_{i=1}^{m} \phi_i \mathbf{tr}\left[\mathbf{D}_{k,i}\mathbf{P}^-_k \right] \right\}_j}{-\mathbf{K}_k\left[\tilde{\mathbf{y}}_k - \mathbf{h}(\hat{\mathbf{x}}^-_k\ ,\ k) \right]_j} \tag{33}$$

or alternatively using the definition of $\mathbf{Y}$ and $\mathbf{Z}$,

$$\mathbf{\Gamma}_{jj} \propto \frac{\mathbf{Y}_j}{\mathbf{Z}_j} \tag{34}$$

Finally, introducing a scale factor $f_{r,j}$ to write the proportional relationship as an equality, and only considering the absolute values of $\mathbf{Y}$ and $\mathbf{Z}$, gives rise to

$$\mathbf{\Gamma}_{jj} = f_{r,j}\frac{|\mathbf{Y}_j|}{|\mathbf{Z}_j|} \tag{35}$$

which, in terms of the diagonal matrix $\boldsymbol{\beta} = \text{diag}\begin{bmatrix}\mathbf{\Gamma}_{11} & \dots & \mathbf{\Gamma}_{nn}\end{bmatrix}$, reads

$$\boldsymbol{\beta}_{jj} = 1 - f_{r,j}\frac{|\mathbf{Y}_j|}{|\mathbf{Z}_j|} \tag{36}$$

Note that, if needed, the scaling term $f_{r,j}\frac{|\mathbf{Y}_j|}{|\mathbf{Z}_j|}$ is manually restricted to the domain $f_{r,j}\frac{|\mathbf{Y}_j|}{|\mathbf{Z}_j|} \in \begin{bmatrix}0 & 1\end{bmatrix}$ so that $\beta_{jj} \in \begin{bmatrix}0 & 1\end{bmatrix}$. If $\mathbf{Z}_j$ equals zero (because the sensor and the expected measurements are the same, or the Kalman gain is zero), $\mathbf{\Gamma}_{jj}$ is set to 1.

In Equation (36), the scale factor $f_{r,j}$ can also act as a variable that adjusts the impact of the nonlinearity metric to the problem in question. As per experiments done with this partial-update approach, an adaptive scale factor $f_{r,j}$ (for the j^{th} partially updated state) involving the measurement residual and the state uncertainty covariance was found conveniently defined as

$$f_{r,j} = \frac{\sigma_{k,j}}{\sigma_{o,j}}\frac{\mathbf{tr}(\mathbf{H}_k\mathbf{P}_k\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k)}{\mathbf{tr}(\mathbf{R}_k)} \tag{37}$$

The reason for including the ratio of the traces is to limit the second-order terms' negative effects commensurate to the current system uncertainty. To account for the impact of initial filter uncertainties, the ratio of the standard deviation at time k for the j^{th} state, $\sigma_{k,j} = \sqrt{(\mathbf{P}_{jj})_k}$, to the corresponding initial value, $\sigma_{o,j}$, is also included in $f_{r,j}$.

B. Nonlinear Covariance-aware partial-update

This section presents an alternative way of selecting the partial-update weights online, which we called nonlinear dynamic covariance-aware partial-update, DC for short. Paralleling the previous method that monitors the EKF2 second-order effects, the method proposed in this section monitors Kalman second-order covariance terms. As before, the aim is that the partial update is reduced when the high-order to first-order terms ratio is significant and increased when the first-order terms are dominant.

To obtain the expressions for the β selection using the covariance-aware method, first, consider the covariance measurement update expression for the second-order Kalman filter [3],

$$\mathbf{P}^+_{k(2)} = \mathbf{P}^-_{k(2)} - \mathbf{P}^-_{k(2)}\mathbf{H}_k^{\mathrm{T}}(\mathbf{H}_k\mathbf{P}^-_{k(2)}\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k + \mathbf{\Lambda}_k)^{-1}\mathbf{H}_k\mathbf{P}^-_{k(2)} \tag{38}$$

Next, to relate the partial-update percentages, $\mathbf{\Gamma}$, to the second-order covariance terms, the expanded covariance partial-update expression for (15) is obtained

$$\mathbf{P}_k{}^{++} = \mathbf{P}_k^+ + \mathbf{\Gamma}\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}}(\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k)^{-1}\mathbf{H}_k\mathbf{P}_k^-\mathbf{\Gamma} \tag{39}$$

where Equations (6) and (7) were substituted into Equation (15). Further, Equation (39) can be written as a function of the prior state covariance as,

$$\begin{aligned}\mathbf{P}_k{}^{++} = \mathbf{P}_k^- - \mathbf{K}_k\mathbf{H}_k\mathbf{P}_k^- \; + \\ \mathbf{\Gamma}\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}}(\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k)^{-1}\mathbf{H}_k\mathbf{P}_k^-\mathbf{\Gamma}\end{aligned} \tag{40}$$

and replacing the Kalman gain with

$$\mathbf{K}_k = \mathbf{P}_k^-\mathbf{H}_k(\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + \mathbf{R})^{-1} \tag{41}$$

leads to

$$\begin{aligned}\mathbf{P}_k{}^{++} = \mathbf{P}_k^- - \mathbf{P}_k^-\mathbf{H}_k(\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + \mathbf{R})^{-1}\mathbf{H}_k\mathbf{P}_k^- \; + \\ \mathbf{\Gamma}\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}}(\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k)^{-1}\mathbf{H}_k\mathbf{P}_k^-\mathbf{\Gamma}\end{aligned} \tag{42}$$

Before attempting to relate the posterior covariance from Equation (42) to (38), two more manipulations are performed. First, Equation (38) is re-written so that the residual covariance term $(\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k)^{-1}$ appears by itself. This is accomplished by applying the matrix inversion lemma to the parenthetical of Equation (38) [19]:

$$\begin{aligned}&(\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k + \mathbf{\Lambda}_k)^{-1} \; = \\ &(\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k)^{-1} \; - \\ &(\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k)^{-1}\mathbf{\Lambda}_k[(\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k)^{-1}\mathbf{\Lambda}_k + \mathbf{I}]^{-1} \; * \\ &(\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k)^{-1}\end{aligned} \tag{43}$$

Second, this expression is substituted into the EKF2 update covariance of Equation (38), which results in

$$\begin{aligned}&\mathbf{P}_{k(2)}{}^{+} = \\ &\mathbf{P}^-_{k(2)} - \mathbf{P}^-_{k(2)}\mathbf{H}_k(\mathbf{H}_k\mathbf{P}^-_{k(2)}\mathbf{H}_k^{\mathrm{T}} + \mathbf{R})^{-1}\mathbf{H}_k\mathbf{P}^-_{k(2)} \; + \\ &\mathbf{P}^-_{k(2)}\mathbf{H}_k^{\mathrm{T}}(\mathbf{H}_k\mathbf{P}^-_{k(2)}\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k)^{-1}\mathbf{\Lambda}_k[(\mathbf{H}_k\mathbf{P}^-_{k(2)}\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k)^{-1}\mathbf{\Lambda}_k + \mathbf{I}]^{-1} \; * \\ &(\mathbf{H}_k\mathbf{P}^-_{k(2)}\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k)^{-1}\mathbf{H}_k\mathbf{P}^-_{k(2)}\end{aligned} \tag{44}$$

Now, by doing a term-by-term comparison of the partial-update expression of Equation (42), and the EKF2 update for the error state covariance of Equation (44), the following relationship between second-order covariance effects and partial-update terms can be established,

$$\begin{aligned}&\mathbf{\Gamma}\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}}(\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k)^{-1}\mathbf{H}_k\mathbf{P}_k^-\mathbf{\Gamma} \sim \\ &\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}}(\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k)^{-1}\mathbf{\Lambda}_k[(\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k)^{-1}\mathbf{\Lambda}_k + \mathbf{I}]^{-1} \; * \\ &(\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k)^{-1}\mathbf{H}_k\mathbf{P}^-\end{aligned} \tag{45}$$

where the symbol $\sim$, is to indicate that the terms are related, and the assumption that the partial-update estimates are similar to those of the EKF2 has been used. For the sake of clarity, Equation (45) can be compactly written as

$$\mathbf{\Gamma}\delta\mathbf{P}_k^-\mathbf{\Gamma} \sim \mathbf{K}_k\mathbf{\Lambda}_k[(\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k)^{-1}\mathbf{\Lambda}_k + \mathbf{I}]^{-1}\mathbf{K}_k{}^{\mathrm{T}} \tag{46}$$

where $\delta\mathbf{P}_k^- = \mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}}(\mathbf{H}_k\mathbf{P}_k^-\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k)^{-1}\mathbf{H}_k\mathbf{P}_k^-$. To further simplify relation (46), the matrix on the right is condensed into a matrix called $\mathbf{N}$; now Equation (46) reads

$$\mathbf{\Gamma}\delta\mathbf{P}_k^-\mathbf{\Gamma} \sim \mathbf{N}_k \tag{47}$$

Recalling that the desire is to select $\mathbf{\Gamma}_{jj}$ proportional to second-order effects, it is proposed to select the $\mathbf{\Gamma}_{jj}$ values by a straight element-by-element comparison of the diagonal elements of matrix $\mathbf{\Gamma}\delta\mathbf{P}_k^-\mathbf{\Gamma}$ and $\mathbf{N}$ at any time k. This leads to the proportionality relationship

$$\delta\mathbf{P}_{jj}^-\gamma_j^2 \propto \mathbf{N}_{jj} \tag{48}$$

or since $\delta\mathbf{P}^-$ and $\mathbf{N}$ are positive semi-definite,

$$\gamma_j \propto \sqrt{\frac{\mathbf{N}_{jj}}{\delta\mathbf{P}_{jj}^-}} \tag{49}$$

As for the nonlinearity-aware method, a scale factor $f_{c,j}$ is introduced to account for measurement residual covariance effects as,

$$\mathbf{\Gamma}_{jj} = \gamma_j = f_{c,j}\sqrt{\frac{\mathbf{N}_{jj}}{\delta\mathbf{P}_{jj}^-}} \tag{50}$$

or equivalently

$$\boldsymbol{\beta}_{jj} = \beta_j = 1 - f_{c,j}\sqrt{\frac{\mathbf{N}_{jj}}{\delta\mathbf{P}_{jj}^-}} \tag{51}$$

The scale factor $f_{c,j}$ used for the covariance-aware method is the same as the one used for the nonlinearity-aware method,

$$f_{c,i} = \frac{\sigma_{k,i}}{\sigma_{o,i}}\frac{\mathbf{tr}(\mathbf{H}_k\mathbf{P}_k\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k)}{\mathbf{tr}(\mathbf{R}_k)} \tag{52}$$

Note that the value for β_i is constrained to $\beta_i \in \begin{bmatrix}0 & 1\end{bmatrix}$.

IV. Numerical Example: The Re-entry Falling Body

The following filter example estimates the altitude, x_1 (in meters), vertical velocity, x_2 (in meters per second), and constant ballistic parameter, x_3 (with units of 1/meter), of a body re-entering the Earth's atmosphere from high altitude and velocity. The discretized nonlinear dynamics, adopted from [21], are given by

$$x_{1(k)} = x_{1(k-1)} + x_{2(k-1)}\Delta t + w_1 \tag{53a}$$

$$x_{2(k)} = x_{2(k-1)} + (e^{\frac{-x_{1(k-1)}}{k_p}}x_{2(k-1)}^2 x_{3(k-1)} - g)\Delta t + w_2 \tag{53b}$$

$$x_{3(k)} = x_{3(k-1)} + w_3 \tag{53c}$$

For this example, the available observations are range measurements h, which are modeled via

$$h(x_1) = \sqrt{d^2 + (x_1 - h_0)^2} + \mathbf{v}_k \tag{54}$$

Here, Δt is the integration step, $k_p = 6.1 \times 10^3\,\mathrm{m}$ relates the air density with the altitude, $g = 9.81\,\mathrm{m/s^2}$ is the acceleration due to gravity, $d = 3 \times 10^4\,\mathrm{m}$ is the horizontal distance from the measuring device to the vertical line traced by the falling body, $h_0 = 3 \times 10^4\,\mathrm{m}$ is the altitude of the measuring device from ground level, and $\mathbf{v}_k$ is zero-mean Gaussian white noise associated with the measurement. The described falling body is pictured in Figure 1. For all simulations presented in this section, the true initial conditions were set to $\mathbf{x}_0 = \begin{bmatrix}100000\text{ m} & -5000\text{ m/s} & 0.003\text{ 1/m}\end{bmatrix}^T$. Initial 1σ uncertainties were set to $\boldsymbol{\sigma} = \begin{bmatrix}10000 & 500 & 0.03\end{bmatrix}$, with appropriate units, and the measurement noise was set to $\mathbf{R} = 1000\ m^2$. As in [21], the process noise is set to zero to avoid masking linearization errors; this is, $\mathbf{w} = [w_1, w_2, w_3]^{\mathrm{T}} = \mathbf{0}$, and $\mathbf{Q} = \mathbf{E}[\mathbf{w}\mathbf{w}^{\mathrm{T}}] = \mathbf{0}$. Note that the filter initial uncertainties referred to as 1σ and its multiples are intended to exercise the filter's capabilities as presented in what follows. Also note that for this example, the partial update is only applied to the ballistic coefficient because it is the state that introduces difficulties when estimated via a conventional EKF.

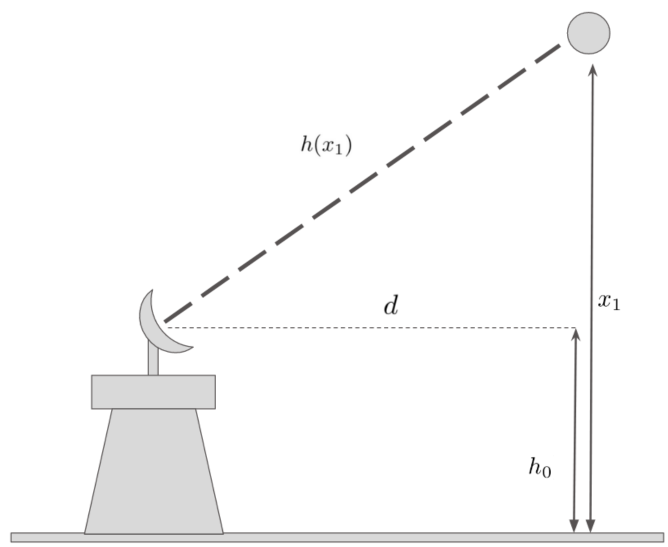

Fig. 1. Falling body diagram.

A. Simulation results

In Figure 2, the states' error histories for the EKF and the static partial-update are shown for a single typical run. The intention of this figure is to demonstrate that at the exercised (1.1σ) level of uncertainty, the EKF is already inconsistent. On the other hand, the partial-update filter that uses a manually tuned (static) update on the ballistic coefficient is consistent as all three states' errors remained within their 3σ bounds. Note that for initial errors smaller than 1σ, the EKF was observed to be consistent.

The results for the nonlinearity-aware and nonlinear covariance-aware partial-update filters and static partial-update, for the same initial conditions used in the run of Figure 2, are plotted in Figure 3. There are two main observations from this figure. First, the nonlinearity-aware methods achieve performance at the level of the finely and manually tuned partial-update filter. And second, the nonlinearity-aware method is able to achieve the lowest uncertainty among the

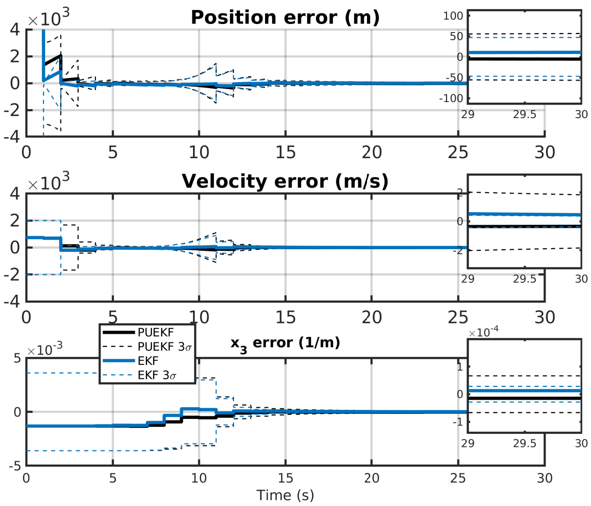

Fig. 2. Partial-update and nominal EKF filters. An initial error of 1.1σ. Partial-update used a $\boldsymbol{\beta} = \text{diag}\begin{bmatrix}1 & 1 & 0.75\end{bmatrix}$. Conventional EKF is not consistent.

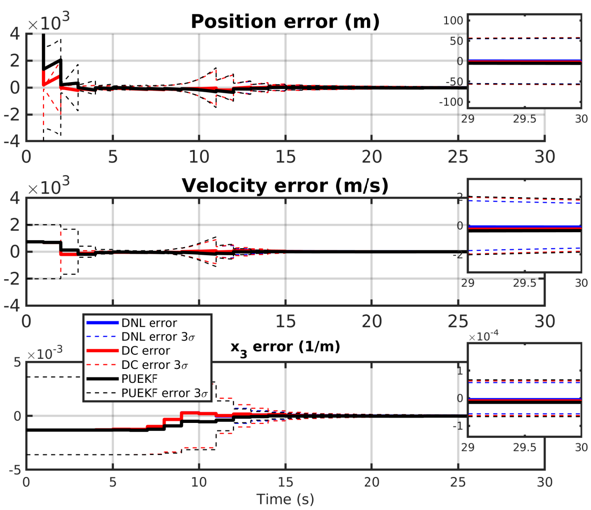

Fig. 3. Dynamic methods (DNL and DC), and the static partial-update using $\boldsymbol{\beta} = \text{diag}\begin{bmatrix}1 & 1 & 0.75\end{bmatrix}$. An initial error of 1.1σ. All filters state errors are consistent.

three methods while maintaining its state errors within their corresponding 3σ bounds. The nonlinearity-aware method achieves the lowest overall uncertainty because it is able to leverage better occasions where a full update is suitable. The covariance-aware method, although it also acts opportunistically when updating, it applies more conservative updates overall. Finally, the static partial-update is observed to achieve more confident bounds than the covariance-aware method, even when it does not monitor nonlinearities. However, the covariance-aware technique dynamically selects the partial-update weights.

Figure 4 displays the β weight applied to the ballistic coefficient for the run shown in Figure 3. From this β's histories, one

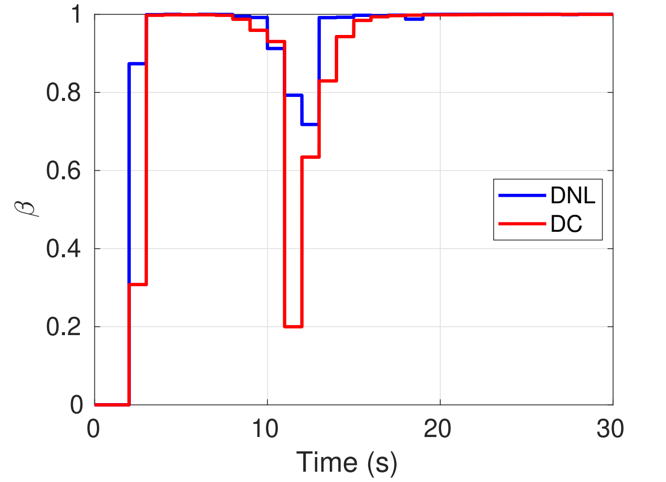

Fig. 4. Partial-update percentages (β) histories for the nonlinearity and covariance-aware methods. Initial uncertainty 1.1σ.

can confirm that the covariance-aware method applies more conservative updates in contrast with the nonlinearity-aware method. Moreover, it can be seen that both dynamic methods are able to detect the high nonlinearities at time $t \approx 11$ secs as both lowered the update percentage for the ballistic coefficient.

A second experiment used an initial condition of 1.5σ to exercise the filter further. The results of a single run using initial errors of 1.5σ, are shown in Figure 5. These results are similar to those obtained for a lower initial error. Although for this specific run, the covariance-aware method incurs larger errors than the other two approaches, it still manages the initial uncertainties and nonlinearities at the level of a finely-tuned static partial update. The corresponding partial-update percentages (β) for the nonlinearity-based methods are shown in Figure 6.

Note from Figure 4 and Figure 6, that the filter percentage selection methods capture two key instants where β is to be significantly varied: at time $t = 1s$ when the first measurement is assimilated, and at time $t = 11s$ after the filter regains information (after losing observability due to range sensor and body alignment).

Although the nonlinearity-based methods perform at the level of the static partial-update approach, one should consider that when using the nonlinearity-based methods tuning or experimentation with the partial-update percentage is not required.

Additionally, 1000 Monte Carlo runs were executed to show that the nonlinearity-based techniques' performance is not specific to a single random draw. The Monte Carlo runs for the nonlinearity-aware and covariance-aware are displayed in Figure 7 and Figure 8, respectively. For completeness, Monte Carlo runs for the static partial-update are included in Figure 9.

In Figures 7, 8, and 9, single filter runs are shown in thin-colored lines, filter-estimated standard deviation are in a solid thick line, and the actual standard deviation is the thick

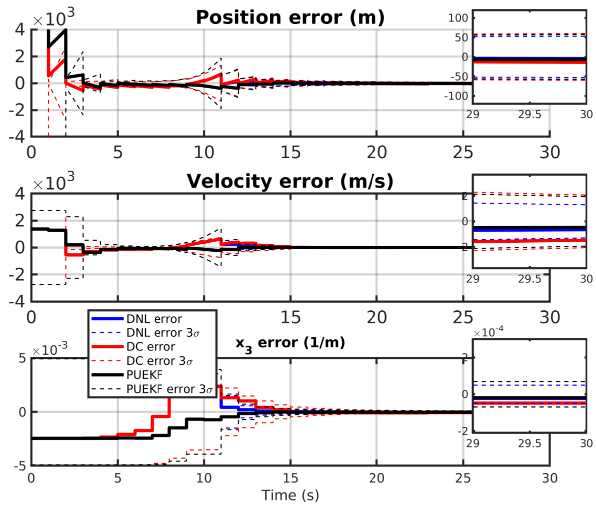

Fig. 5. Dynamic methods (DNL and DC), and the static partial-update using $\boldsymbol{\beta} = \text{diag}\begin{bmatrix}1 & 1 & 0.75\end{bmatrix}$. An initial error of 1.5σ. All filters state errors are consistent.

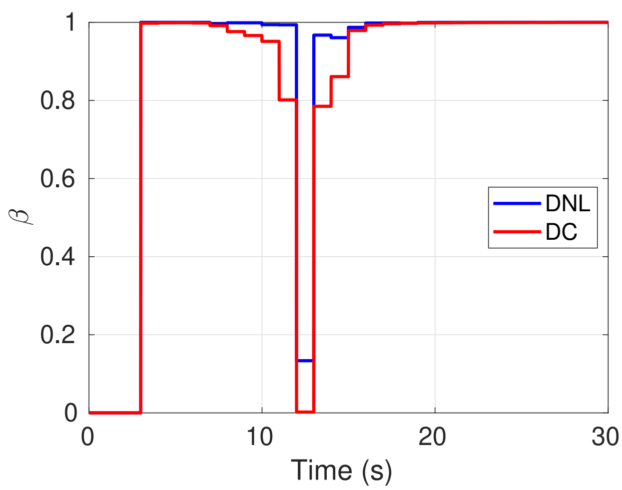

Fig. 6. Partial-update percentages (β) histories for the nonlinearity and covariance aware methods. Initial uncertainty 1.5σ.

dashed line. All three filters used the same initial seed and uncertainties indicated in the problem description. From these figures, it is apparent that both nonlinearity-aware methods incur less error than the static partial-update filter. Furthermore, the nonlinearity-aware filters appear more consistent than the partial-update filter as their estimated uncertainties match the actual sampled uncertainty better. It is worth noting, that even though the consistency of the nonlinearity-aware filters is not perfect, it should be recalled that the filter is still a linear filter and that the conventional EKF was not functional under the presented scenario.

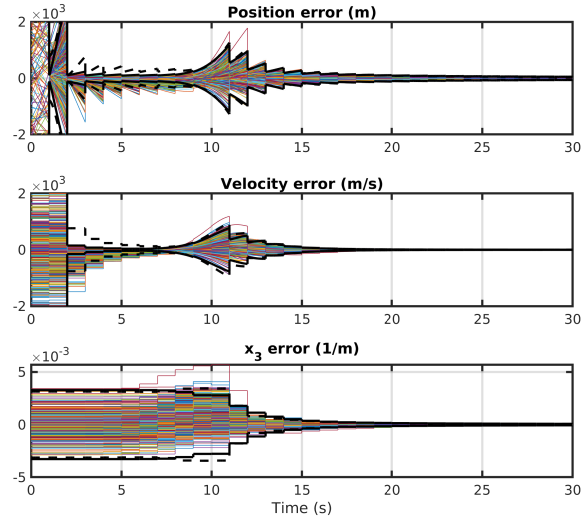

Fig. 7. 1000 Monte Carlo runs of the nonlinearity-aware method along with the actual $\pm 3\sigma$ bounds from all runs (dashed line) and from the filter (solid line).

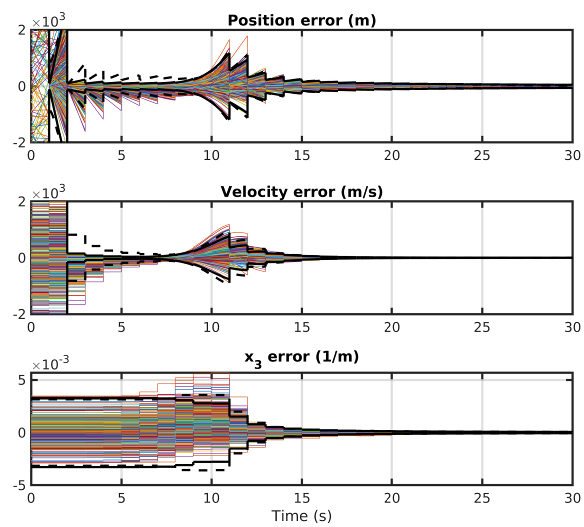

Fig. 8. 1000 Monte Carlo runs of the covariance-aware method along with the actual $\pm 3\sigma$ bounds from all runs (dashed line) and from the filter (solid line).

V. Conclusion

Two methods for selecting the partial-update filter percentages were presented. Both methods were based on directly monitoring nonlinearity metrics to appropriate update percentages such that the Kalman update is limited when second-order terms are comparable to first-order terms. The proposed nonlinearity-aware techniques did not require manual tuning. However, they achieved comparable performance to a finely-tuned static partial-update. This tuning-free characteristic can be advantageous, especially when the size of the state vector is large. Furthermore, simulated Monte Carlo runs showed that filters using the nonlinearity-aware methods incurred in less estimation error and had more appropriate covariances

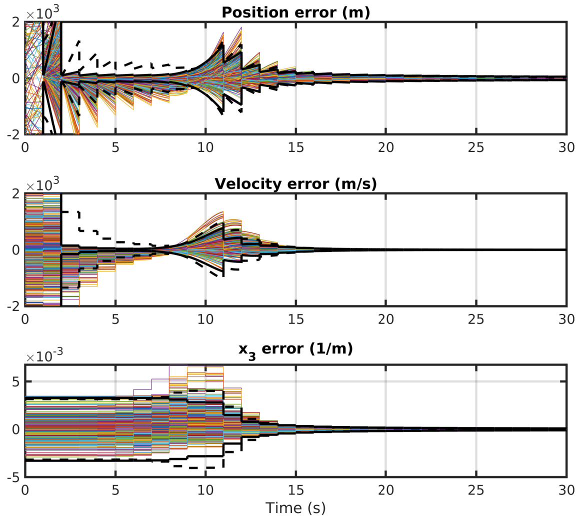

Fig. 9. 1000 Monte Carlo runs of the static partial-update filter with $\boldsymbol{\beta} = \text{diag}\begin{bmatrix}1 & 1 & 0.75\end{bmatrix}$ along with the actual $\pm 3\sigma$ bounds from all runs (dashed line) and from the filter (solid line).

than the static partial-update filter. This was the case because the nonlinearity-aware methods are capable of limiting their updates on highly-nonlinear regimes, where the Kalman filter equations sub-optimality is exacerbated. Finally, it is worth mentioning that the proposed metrics and methods for online partial-update weights (β) selection mostly use information computed within the Kalman filter. Therefore, they do not over-specialize the filter, retain the EKF structure, and facilitate their incorporation into any standard Kalman filter implementation.

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