Fractional Thermoelasticity: a Review

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Fractional Thermoelasticity: A Review

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Abstract

In the present article, a review of fractional order thermoelastic models that look very promising for future development of fractional order theories. We have presented a short introduction to fractional calculus as a theory of integration and differentiation of non-integer order. The recently developed fractional order thermoelastic models are described with their basic mathematical formulation and characteristic features of these models are illustrated. Also, the paper concludes with a discussion on the future potential of the use of fractional order theory of thermoelasticity for the analysis of thermodynamical interactions in solid.

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1 Introduction

Thethermoelasticity theory is a combination of elasticity and heat conduction theories. It is related to the impact of heat on the deformation of an elastic medium and the inverse impact of the deformation on the thermal condition of the considered medium. Thermal stress is produced when the time rate of variation of a heat source in a medium or the time rate of variation of thermal boundary conditions on a medium is compared with the structural oscillation characteristics. In this condition, the solutions to the problem for the are as of temperature and stresses should be derived via the thermoelasticity coupled equations. The classical uncoupled theory of thermoelasticity predicts two phenomena not compatible with physical observations. First, the equation of heat conduction of this theory does not contain any elastic terms; second, the heat equation is of a parabolic type, predicting infinite speeds of propagation for heat waves.

In 1956, Biot M. A. [1] formulated the theory of coupled thermoelasticity to eliminate the paradox inherent in the classical uncoupled theory that elastic changes have no effect on the temperature. The heat equations for both theories are of the diffusion type predicting infinite speeds of propagation for heat waves contrary to physical observations. In 1967, Lord and Shulman [2] introduced the theory of generalized thermoelasticity with one relaxation time for the special case of an isotropic body. This theory was extended by Dhaliwal and Sherief [3] to include the anisotropic case. In this theory a modified law of heat conduction including both the heat flux and its time derivative replaces the conventional Fouriers law. The heat equation associated with this theory is hyperbolic and hence eliminates the paradox of infinite speeds of propagation inherent in both the uncoupled and the coupled theories of thermoelasticity. The second generalization to the coupled theory of elasticity is known as the theory of thermoelasticity with two relaxation times or the theory of temperature rate dependent thermoelasticity.

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In 1969, Fox [4] introduced the generalisation of thermoelasticity is proposed which arises from a physically motivated modification of Fouriers law of heat conduction. Constitutive equations are postulated which are valid for finite deformations and temperature variations and these are reduced to canonical form by the usual techniques of non-linear continuum mechanics. In 1972, Green and Lindsay [5] developed the theory of generalized thermoelasticity with two relaxation times which is based on a generalized inequality of thermodynamics. This theory does not violate Fouriers law of heat conduction when the body under consideration has a center of symmetry. In this theory both the equations of motion and of heat conduction are hyperbolic but the equation of motion is modified and differs from that of coupled thermoelasticity theory.

2 Historical Review

Fractional calculus is an extension of ordinary calculus with more than 300 years of history. Fractional calculus was initiated by Leibniz and LHospital as a result of a correspondence which lasted several months in 1695. In that year, Leibniz wrote a letter to LHospital raising the following question [6] Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders ? “LHopital was somewhat curious about the above question and replied by another simple one to Leibniz: What if the order will be 1/2 ?”. Leibniz in a letter dated September 30, 1695, replied: “It will lead to a paradox, from which one day useful consequences will be drawn”.

That date is regarded as the exact birthday of the fractional calculus. The issue raised by Leibniz for a fractional derivative (semi-derivative, to be more precise) was an ongoing topic in decades to come [6, 7]. Following LHopitals and Liebnizs first inquisition, fractional calculus was primarily a study reserved for the best mathematical minds in Europe. Euler [7], wrote in 1730: “When n is a positive integer and p is a function of $x, p = p(x)$, the ratio of $d^n p$ to dx^n can always be expressed algebraically. But what kind of ratio can then be made if n be a fraction ?”.

A lot of contributions to the theory of fractional calculus up to the middle of the 20th century, of famous mathematician are known: Lagrange in 1772, Laplace in 1812, Lacroix in 1819, Fourier in 1822, Abel in 1823, Liouville in 1832, Riemann in 1847, Green in 1859, Holmgren in 1865, Grunwald in 1867, Letnikov in 1868, Sonini in 1869, Laurent in 1884, Nekrassov in 1888, Krug in 1890, Heaviside in 1892, Weyl in 1919, and many others. During the 19th century, the theory of fractional calculus was developed primarily in this way, trough insight and genius of great mathematicians. Namely, in 1819 Lacroix gave the correct answer to the problem raised by Leibnitz and LHospital for the first time, claiming that $d^{1/2}x/dx^{1/2} = 2\sqrt{x/\pi}$.

3 Literature Review

In 1995, Caputo [8] have employed fractional order derivatives for the description of viscoelastic materials and they have successfully established the relation between the fractional order derivatives and the linear theory of viscoelasticity. They also obtained a very good agreement with the experimental results successfully. Some applications of fractional calculus to various problems of mechanics of solids are reviewed in the book of Mainardi [9]. In the last few years, fractional calculus theory has been employed successfully in theories of thermoelasticity and several models of fractional order generalized thermoelasticity are established by many authors.

In 2004, Povstenko [10] made a survey of non-local generalization of the Fouriers law and presented a problem of thermoelasticity that uses fractional heat conduction equation. In 2008, Povstenko [11] solved the time-fractional radial diffusion in a sphere. The Caputo fractional derivative of the order $0 < \alpha < 2$ is used. The Laplace and finite sin-Fourier transforms are employed. The solution is written in terms of theMittagLeffler functions. In 2009, Povstenko [12] discussed heat conduction problem with time and space fractional derivatives for the classical Fourier law generalizations.

In 2010, Povstenko [13] investigated thermal stresses in the framework of a quasistatic uncoupled theory of thermoelasticity based on heat conduction equation with a time fractional derivative in an infinite medium. In 2010, Sherief et. al. [14] introduced the fractional order theory of thermoelasticity by using the methodology of fractional calculus, proved uniqueness theorem and derived variational principle and reciprocity theorem. In 2010, Youssef [15] formulated the theory of fractional order generalized thermoelasticity by introducing the Riemann-Liouville fractional integral operator into the generalized heat conduction equation, proved uniqueness theorem and solved one dimensional problem.

In 2010, Youssef and Al-Lehaibi [17] constructed a mathematical model of an elastic material with constant parameters fills the half-space and the governing equations will be taken into the context of the fractional order generalized thermoelasticity theory. The medium is assumed initially quiescent and Laplace transforms and state-space techniques will be used to obtain the general solution for any set of boundary conditions. The general solution obtained is applied to a specific problem of a medium subjected to ramp-type heating and traction free.

In 2010, Jiang and Xu [18] obtained the time fractional Fourier law is obtained from fractional calculus. According to the fractional Fourier law, a fractional heat conduction equation with a time fractional derivative in the general orthogonal curvilinear coordinate system is built. The fractional heat conduction equations in other orthogonal coordinate systems are readily obtainable as special cases. The fractional heat conduction equation in the case $0 < \alpha < 1$ interpolates the standard heat conduction equation $\alpha = 1$ and the Localized heat conduction equation $\alpha \to 0$. In 2011, Ezzat [19] developed the a new model of the magneto-thermoelasticity theory has been constructed in the context of a new consideration of heat conduction with fractional derivative. A one-dimensional application for a conducting half-space of thermoelectric elastic material, which is thermally shocked in the presence of a magnetic field, a conclusion about the new theory of magneto-thermoelasticity has been constructed.

In 2011, Ezzat [22] constructed a new mathematical model of thermoelectric modification of the heat-conduction equation from diffusive theory in the context of a new consideration of heat conduction with fractional orders. This model is applied to Stokes first problem for a conducting fluid with heat sources. The modification of the heat-conduction equation from diffusive to a wave type may be affected either by a microscopic consideration of the phenomenon of heat transport or in a phenomenological way by modifying the classical Fourier law of heat conduction.

In 2011, Povstenko [23] introduced the generalized Cattaneo-type equations with time-fractional derivatives. The corresponding theory of thermal stresses is formulated. The proposed theory, on the one hand, interpolates the theory of Lord and Shulman and thermoelasticity without energy dissipation of Green and Naghdi and, on the other hand, generalizes theory of thermal stresses based on the fractional heat conduction equation. The fundamental solution to the nonhomogeneous fractional telegraph equation as well as the corresponding stresses are obtained in one-dimensional and axisymmetric cases.

In 2012, Povstenko [24] solved the time-fractional diffusion-wave equation is considered in a sphere in the case of three spatial coordinates r, μ, and ϕ. The Caputo fractional derivative of the order$0 < \alpha \leq 2$ is used. The solution is found using the Laplace transform with respect to time t, the finite Fourier transform with respect to the angular coordinate ϕ, the Legendre transform with respect to the spatial coordinate ϕ, and the finite Hankel transform of the order $n + 1/2$ with respect to the radial coordinate r. In 2012, Sherief et. al. [26] investigate the methodology of fractional calculus is used to formulate a new generalized fractional of the Hodgkin and Huxley model for action-potential of a neuron. The Caputo fractional derivative is used in this model. A numerical algorithm based on the predictor corrector method is used to obtain the solution of the model. The results showed that as the fractional derivative order tends to unity the original model is retained. Every change in the fractional order gave a new response for membrane voltage.

In 2012, Youssef [27] discussed the two-dimensional thermal shock problem of fractional order generalized thermoelasticity. Also, considered a half-space filled with an elastic material, which has constant elastic parameters. The governing equations are taken in the context of the fractional order generalized thermoelasticity theory. In 2012, Gaikwad et. al. [28] solved the nonhomogeneous thermoelastic problem of thermal deflection with internal heat generation. In 2013, Gaikwad [29] analyzed the thermoelastic deformation of a thin hollow circular disk due to partially distributed heat supply.

In 2013, Ezzat et. al. [30] developed the new fractional relaxation operator using the methodology of fractional calculus. The governing coupled fractional differential equations in the frame of the thermo-viscoelasticity with fractional order heat transfer are applied to the one-dimensional problem with heat sources. In 2013, Povstenko [31] solved the problem of fractional heat conduction in a composite medium consisting of a spherical inclusion $(0 < r < R)$ and a matrix $(R < r < \infty)$ being in perfect thermal contact at $r = R$. The heat conduction in each region is described by the time-fractional heat conduction equation with the Caputo derivative of fractional order $0 < a \leq 2$ and $0 < \beta \leq 2$, respectively.

In 2014, Hussain [32] solved an infinitely long solid circular cylinder, whose lateral surface is traction free and subject to a known surrounding temperature. The problem is in the context of the fractional order thermoelasticity theory. The medium is assumed to be initially quiescent. The solution is obtained by a direct approach without the customary use of potential functions.

In 2015, Povestenko [33] is devoted to the time- and space-nonlocal generalization of Fouriers law, corresponding generalization of the heat conduction equation and formulation of the associated theories of fractional thermoelasticity. This book also presents a picture of state-of-art of fractional thermoelasticity and some problems for the time-fractional heat conduction equation.

In 2015, Caputo and Fabrizio [34] developed the new definition of fractional derivative with a smooth kernel which takes on two different representations for the temporal and spatial variable. The first works on the time variables; thus it is suitable to use the Laplace transform. The second definition is related to the spatial variables, by a non-local fractional derivative, for which it is more convenient to work with the Fourier transform.

In 2015, Raslan [35] solved the fractional order theory of thermoelasticity is applied to the two-dimensional problem of a thick plate whose lower and upper surfaces are traction free and subjected to a given axi-symmetric temperature distribution. In 2015, Zenkour and Abouelregal [36] proposed, a theory of two-temperature generalized thermoelasticity is constructed in the context of a new consideration of heat conduction with fractional orders. The obtained general solution is applied to a specific problem of a medium, semi-infinite solid considered to be made of a homogeneous thermoelastic material. The bounding plane surface of the medium is being subjected to a non-Gaussian laser pulse.

In 2016, Gaikwad [37] proposed a two-dimensional steady-state temperature distribution of a thin circular plate due to uniform internal energy generation. In 2016, Ezzat et. al. [39] solved the three-dimensional model of thermoelasticity with fractional order heat transfer is established. The resulting non-dimensional coupled equations together with the Laplace and double Fourier transforms techniques are applied to a half space which assumed to be traction free and subjected to a thermal shock that is a function of time. In 2016, Tripathi et. al. [40] determining the thermoelastic displacement, stress, and temperature in a thick circular plate of finite thickness and infinite extent whose lower and upper surfaces are traction free, subjected to a given axisymmetric temperature distribution. The problem is formulated in the context of fractional order thermoelasticity theory with finite wave speeds.

In 2017, Warbhe et. al. [41] solve the time nonlocal generalization of the classical Fourier law with the Long-tail power kernel can be interpreted in terms of fractional calculus and leads to the time fractional heat conduction equation. In 2018, Povstenko and Kyrylych [42] deals the symmetric stress distribution in an infinite isotropic plane with a line crack, in which surfaces are exposed to the heat flux loading is considered in the framework of fractional thermoelasticity. In 2019, Khobragade and Lamba [43] presents the Magneto-thermodynamic Response of Orthotropic Solid Cylinder in context of fractional order theory. The Laplace transform and finite MarchiZgrablich transform techniques have been used to analyze the thermal behavior of solid cylinder.

In 2019, Gaikwad [44] discussed the axi-symmetric thermoelastic stress analysis of a thin circular plate due to heat generation. In 2019, Gaikwad and Khavale [45] deals with fractional heat conduction problem in a thin hollow circular disk and its thermal deflection. In 2019, Mittal and Kulkarni [46] deals with an application of fractional order thermoelasticity in association with two-temperature theory. The fractional heat conduction model has been proposed to investigate the thermal variations within the bounded spherical region. The corresponding heat conduction equation has been derived in the context of the generalized two-temperature theory of fractional thermoelasticity.

In 2020, Khavale and Gaikwad [47] discussed the generalized theory of magneto-thermoviscoelastic spherical cavity problem under fractional order derivative using state space approach. In 2020, Ezzat [48] solved the fractional model of thermo-viscoelasticity theory with variable thermal conductivity and rheological properties of the volume is considered to investigate bio-thermo-mechanics behavior in living tissue. The model is applied to a problem of cancerous layer with arbitrary thickness and its outer surface traction free. The bounding plane of the cancerous tissue is subjected to two different types of thermal loading; a non-Gaussian laser beam and harmonic heating. Several authors studied a number of problems by employing the above mentioned theories of fractional order generalized thermoelasticity in solid medium. Some of them are found in the references [49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66]

4 Fractional order derivatives: definition and formulation

♢ The Riemann-Liouville Fractional Differential Operator [67]

Suppose that $\alpha > 0$, $t > a$, α, $a, t \in R$. Then

$$D^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \dfrac{d^n}{dt^n} \int_0^t \dfrac{f(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, & \text{n-1} < \alpha < n \,\epsilon\, N; \\ \dfrac{d^n f(t)}{dt^n}, & \alpha = \text{n} \in N \end{cases} \tag{4.1}$$

is called the Riemann-Liouville fractional derivative of order α.

◇ The Caputo Fractional Differential Operator [67]

Suppose that $\alpha > 0$, $t > a$, α, $a, t \in$ R. Then

$$D^{\alpha} f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^n(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, & \text{n-1} < \alpha < n \, \epsilon \, N; \\ \frac{d^n f(t)}{dt^n}, & \alpha = \text{n} \in N \end{cases} \tag{4.2}$$

is called the Caputo fractional derivative of order α.

◇ The Mittag-Leffler Function [67]

The Mittag-Leffler function is a generalization of the exponential function, first introduced as a one-parameter function by the series [10]

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0, \ \alpha \in R, z \in C. \tag{4.3}$$

Later, the two-parameter generalization Mittag-Leffler function as:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0, \ \alpha, \beta \in R, z \in C. \tag{4.4}$$

5 Fractional order generalized thermoelastic models

1. **Sherief s Model of Fractional Order Generalized Thermoelastic Heat Conduction** [14]:

The heat conduction equation in this model takes the form:

$$q_i + \tau_0 \frac{\partial^{\alpha}}{\partial t^{\alpha}} = -k_{ij} T_{,j} \qquad 0 < \alpha \leq 1. \tag{5.1}$$

2. **Youssef s Model of Fractional Order Generalized Thermoelastic Heat Conduction** [15]:

The heat conduction equation in this model takes the form:

$$q_i + \tau_0 \frac{\partial}{\partial t} q_i = -k_{ij} I^{\alpha-1} T_{,j} \qquad 0 < \alpha \leq 2, \tag{5.2}$$

$$\text{where} \quad I^{\alpha} f(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(x,s) ds.$$

3. **Ezzats Model of Fractional Order Generalized Thermoelastic Heat Conduction** [16]:

The heat conduction equation in this model takes the form:

$$q_i + \frac{\tau_0}{\Gamma(\alpha+1)} \frac{\partial^{\alpha}}{\partial t^{\alpha}} q_i = -k_{ij} T_{,j} \qquad 0 < \alpha \leq 1. \tag{5.3}$$

4. **Fractional Order Two-Temperature Generalized Thermoelastic Heat Conduction Model (El-Karamany and Ezzat)** [20]:

The heat conduction equation in this model takes the form:

$$q_i + \frac{\tau_0}{\Gamma(\alpha+1)} \frac{\partial^{\alpha}}{\partial t^{\alpha}} q_i = -k_{ij} \phi_{,j} \qquad 0 < \alpha \leq 1. \tag{5.4}$$

5. **Fractional Order Generalized Thermoelastic Diffusion Model (Ezzat and Fayik)** [21]:

The equation of mass flux vector in this model takes the form:

$$\eta_j + \frac{\tau_0}{\Gamma(\alpha+1)} \frac{\partial^{\alpha}}{\partial t^{\alpha}} \eta_j = -D P_{,i} \qquad 0 < \alpha \leq 1. \tag{5.5}$$

6. **Fractional Order Generalized Thermoelastic Model with Three-Phase-Lag (Ezzat et al.)** [25]:

The equation of heat conduction in this model takes the form:

$$\left(1+\frac{\tau_q^\alpha}{\Gamma(\alpha+1)}\frac{\partial^\alpha}{\partial t^\alpha}+\frac{\tau_q^{2\alpha}}{\Gamma(2\alpha+1)}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)q_j=\left(\tau_\upsilon^* T_{,j}+k_{ij}\frac{\tau_T^\alpha}{\Gamma(\alpha+1)}\frac{\partial^\alpha}{\partial t^\alpha}+k_{ij}^*\upsilon_{,i}\right) \tag{5.6}$$

$$\text{for } 0<\alpha\leq 1 \text{ where } \qquad \tau_\upsilon^*=k_{ij}+k_{ij}^*\frac{\tau_\upsilon^\alpha}{\Gamma(\alpha+1)}\frac{\partial^{\alpha-1}}{\partial t^{\alpha-1}}.$$

7. **Generalized Thermoelastic Model with Fractional Order Strain (Youssef)**[38]:

The stress-strain relation in this model is given as:

$$\sigma_{ij}=E^*\tau^{*^\alpha}\frac{\partial^\alpha e_{ij}}{\partial t^\alpha}, \qquad 0<\alpha\leq 1. \tag{5.7}$$

6 Summary

The present article is devoted to the development of theories of generalized thermoelasicity with fractional order of derivative and its application to dynamic problems of solid mechanics and structural mechanics. The detailed description of the generalized thermoelastic models involving fractional derivatives is presented along with the definitions of fractional derivative and fractional integral. Theory of thermoelasticity with fractional order time derivatives is a new branch of research.

7 Applications

The review presented in this paper should prove useful for researchers in material science, designers of new materials, low temperature physicists as well as for those working on the development of a theory of hyperbolic thermoelasticity with fractional order. The fractional order strain problem may be applicable in the fields of biomechanics, biomedical problems and skin tissues where knowledge of such changes would enable early diagnostic monitoring for the onset of disease and better assessment of the effectiveness of new drugs or therapies. The results can also be used in nuclear reactors and space technology, which are beneficial to engineers. Regarding this, many pressure vessels and other engineering structures are subjected simultaneously to the action of stress and high temperature. The continual increase in the temperatures of operation has placed great practical importance on the strength of materials at elevated temperatures and the development of materials to cope with this trend.

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