A Practical Approach to the Homotopy Groups of Spheres

Valerii Sopin

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email: VvS@myself.com

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Abstract

In the paper we utilize correspondence between the i-th homotopy group of (r+1)-sphere and the i-th homotopy group of the wedge sum of i (r+1)-spheres based on Hilton's theorem (the homotopy groups of such wedge sums consolidate all information about homotopy groups of spheres). This leads to a practical method for computing the homotopy groups of spheres. Moreover, it reduces the computation of the homotopy groups of (r+1)-sphere to a combinatorial group theory question.

The approach

The homotopy groups of spheres describe how spheres of various dimensions can wrap around each other, i.e. the i-th homotopy group $\pi_i(S^{r+1})$ summarizes the different ways in which the i-dimensional sphere S^i can be mapped continuously into the (r+1)-dimensional sphere S^{r+1}, but it does not distinguish between two mappings if one can be continuously deformed to the other. However, as opposed to other topological invariants, such as homology groups, the homotopy groups are surprisingly complex and difficult to compute (e.g., $\pi_i(S^2)$ is only known for every $i \leq 64$).

Most methods for computing the homotopy groups of spheres are based on spectral sequences [1]. This is usually done by constructing suitable fibrations and taking the associated long exact sequences of homotopy groups. However, the computation of the homotopy groups of S^2 has been reduced to a combinatorial group theory question. The work [2] identifies these homotopy groups as certain quotients of the Brunnian braid groups of S^2.

In this paper we operate with another correspondence: between the i-th homotopy group of S^{r+1} and the i-th homotopy group of the wedge sum of i (r+1)-spheres. Let $T_{k,r}$ be the wedge sum of k (r+1)-spheres, then Corollary 4.10 of [3] gives isomorphism

$$\pi_i(T_{k,r}) \cong \sum_{w=1}^{\infty}(\text{ sum of } Q(w,k) \text{ copies of } \pi_i(S^{wr+1})\ ),$$

where $Q(w,k) = \frac{1}{w}\sum_{d|w}\mu(d)k^{\frac{w}{d}}$ (μ is the classic Möbius function) is a necklace polynomial, counting, for example, number of degree-k irreducible polynomials over GF(2) or number of binary Lyndon words of length k. Note that, for each i, there are only finitely many non-zero terms on the right-hand side, since the sequence $\{w\}$ tends to infinity. Therefore, we reduce the matter to computing the i-th homotopy group of the wedge sum of i (r+1)-spheres.

Freudenthal suspension theorem [4] is worth mentioning here, as it implies that the suspension homomorphism from $\pi_{n+j}(S^n)$ to $\pi_{n+j+1}(S^{n+1})$ is an isomorphism for $n \geq j+2$. Note that $\Sigma^n(X \vee Y)$ is homeomorphic to $\Sigma^n X \vee \Sigma^n Y$ (in fact, $\Sigma S^n \cong S^{n+1}$), where Σ means suspension (or reduced suspension if one prefers since it doesn't matter for well-pointed spaces) and $\vee$ means wedge sum. So, Freudenthal suspension theorem, being stated in terms of the map $\pi_i(X) \to \pi_{i+1}(\Sigma X)$, gives

$$\pi_i(T_{k,r}) \to \pi_{i+1}(T_{k,r+1}),$$

i.e. the groups stabilize.

Moreover, (r+1)-sphere has a natural choice of comultiplication (it is unique) via the wedge sum, i.e. it is the map pinching the equator of ΣS^r to a point, yielding two tangent copies of itself. The map is commutative up to homotopy (which is the reason that the i-th homotopy group is abelian). So, (r+1)-sphere is co-H-space. The last gives

$$\pi_i(T_{k,r+1}) \to \pi_i(T_{k+1,r+1}).$$

In addition, Corollary 4.10 of [3] shows (see the right-hand side) that the groups $\pi_i(T_{k\to\infty,r})$ are malleable, since for $0 < j < n$, any mapping from S^j to S^n is homotopic to a constant mapping, i.e. the homotopy group $\pi_j(S^n)$ is the trivial group.

Hence, according to the above, computing the homotopy groups of wedge sums of the same dimension spheres is not more complicated than computing the homotopy groups of spheres, but it offers new relations. Thus, it should reveal more patterns. Indeed, it is so, since the paper [5] universalises the work [2]. Furthermore, the loop space ΩX, i.e the space of continuous pointed maps from the pointed circle S^1 to X, equipped with the compact-open topology (two loops can be multiplied by concatenation, so the loop space is an A_∞-space; the loop space is dual to the suspension of the same space), provides the following equivalences

$$\pi_i(X) \cong \pi_{i-1}(\Omega X),$$

$$\pi_i(\prod_{j=1}^{k-1} \Omega T_{j,r}) \cong \pi_i(\Omega \mathrm{Conf}(\mathbb{R}^{r+2}, k)),$$

where the configuration space of ordered k-tuples of distinct points in Y is defined as the subspace of Y^k given by $\mathrm{Conf}(Y, k) = \{(y_1, \ldots, y_k\} \mid y_s \neq y_t \text{ for all } s \neq t\}$. Besides, I. James showed [6] that ΩS^n is homotopy equivalent to a CW complex with a single cell in each dimension divisible by $n-1$ (it is important to note that J. Milnor [7] then proved that any loop space is homotopy equivalent to a geometric realization of a simplicial group [8], and so, theoretically speaking, the homotopy groups of any space can be determined as the homology of a Moore chain complex). And the configuration space $\mathrm{Conf}(\mathbb{R}^n, k))$ is homeomorphic to $\mathbb{R}^n \times \mathrm{Conf}(\mathbb{R}^n \setminus \{0\}, k-1)$, where $\{0\}$ is the set with a single point given by the origin in $\mathbb{R}^n$.

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