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Efficient Algorithms for Computation of Equilibrium/Transient Probability Distribution of Finite Markov Chains: Potential Lower Bound on Computational Complexity

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EFFICIENT ALGORITHMS FOR COMPUTATION OF EQUILIBRIUM/TRANSIENT PROBABILITY DISTRIBUTION OF FINITE CONTINUOUS TIME MARKOV CHAINS: POTENTIAL LOWER BOUND ON COMPUTATIONAL COMPLEXITY

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ABSTRACT

In this research paper, efficient algorithms for computation of equilibrium as well as transient probability distribution of arbitrary finite state space Continuous Time Markov Chain are proposed. The effective idea is to solve a structured systems of linear equations efficiently. The algorithms potentially achieve lower bound on the computational complexity (of solving structured system of linear equations arising in the equilibrium/transient analysis of Continuous Time Markov Chains).

1. Introduction:

Markov chains provide interesting stochastic models of natural/artificial phenomena arising in science and engineering. The existence of equilibrium behavior enables computation of equilibrium performance measures. Thus, researchers invested considerable effort in EFFICIENTLY computing the equilibrium probability distribution of Markov chains and thus, the equilibrium performance measures.

Traditionally, computation of transient probability distribution of Continuous Time Markov Chains (CTMCs) was considered to be a difficult open problem. It requires computation of MATRIX EXPONENTIAL associated with the generator matrix of CTMC. Even in the case of finite state space CTMCs, efficient computation of transient probability distribution was considered to be a difficult open problem. In [Rama1], an interesting approach for recursive computation of transient probability distribution of arbitrary finite state space CTMCs was proposed.

It is well known that computation of equilibrium distribution of CTMCs reduces to solution of linear system of equations. The approach proposed in [Rama 2] reduces the computation of transient probability distribution of finite state space CTMCs to solving linear system of equations in the transform (Laplace Transform) domain. Thus, an interesting question that remained deals with efficient solution of such STRUCTURED linear system of equations in the Laplace transform domain. In fact, a more interesting problem is to design an algorithm which meets a LOWER BOUND on the computation of

solution of structured system of linear equations arising in the transient/equilibrium analysis of CTMCs.

This research paper is organized as follows. In Section 2, algorithm for efficient computation of equilibrium probability mass function of Continuous Time Markov Chains (CTMCs) is discussed. In Section 3, algorithm for efficient computation of transient probability mass Function of Continuous Time Markov Chains is discussed. In section 4, we briefly discuss how the proposed algorithm potentially achieves the lower bound on computation of equilibrium/transient Probability Mass Function. The research paper concludes in Section 4.

2. Efficient Computation of Equilibrium Probability Mass Function of Continuous Time Markov Chains:

It is well known that computation of equilibrium probability vector (PMF) $\bar{\pi}$ of a Homogeneous Continuous Time Markov Chain (CTMC) with generator $\bar{Q}$ reduces to the solution of following linear system of equations (i.e. computation of vector $\bar{\pi}$ in the left null space of the generator matrix $\bar{Q}$) i.e.

$$\bar{\pi}\,\bar{Q} \equiv \bar{0}\,.$$

In computational linear algebra, there are efficient algorithms to solve linear system of equations. In fact, "COMPUTATIONALLY OPTIMAL ALGORITHMS" are designed to reduce the computational complexity (in terms of elementary operations such as additions, multiplications etc.). But, from the point of view of computing $\bar{\pi}$, they donot take into account the structure of generator matrix $\bar{Q}$ .Thus, the goal of this research paper is to design efficient algorithm for computing $\bar{\pi}$, taking into account the structure of generator matrix $\bar{Q}$. In fact, we would like to design a COMPUTATIONALLY OPTIMAL ALGORITHM for computation of $\bar{\pi}$ ($in\ terms\ of$ *Computational Complexity*).

We illustrate the essential idea with a Continuous Time Markov chain (CTMC) with 4 states. First consider the generator matrix of a CTMC with 4 states (i.e. 4 x 4 matrix) partitioned using blocks of size 2 x 2 (i.e. there are 4 such 2 x 2 matrices in the generator). Specifically, we have

$$Q = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \qquad where\ \{\, A_{ij};\ i,j\ \in \{1,2\}\,\}\ are$$

$$2\,x\,2\ \ square\ \ matrices\ \ in\ \ the\ \ 4\,x\,4\ \ block\ matrix\ \ Q.$$

Claim: For a positive recurrent (recurrent non-null) CTMC with generator matrix Q,

it is well known that the matrices $\{A_{11}, A_{22}\}$ are non-singular.

Hence, we have that

$$A_{21} = A_{22}\,X \quad with \quad X = A_{22}^{-1}\,A_{21}.$$

Thus, by means of elementary column operations, the generator matrix Q can be converted into the following form i.e.

$$\widetilde{Q} = \begin{bmatrix} \widetilde{A_{11}} & \widetilde{A_{12}} \\ 0 & A_{22} \end{bmatrix} \; i.e.\; block\; upper\; triangular\; matrix.$$

In fact, let $X = [\,f_1 \quad f_2\,]$. Hence, column operations to arrive at $\tilde{Q}$ (from Q) are determined by the column vectors $\{\,f_1 \quad f_2\,\}$. Also, it follows from linear algebra, that the equilibrium probability vector $\bar{\pi}$ is unaffected by such column operations. Hence, with $\bar{\pi} = [\,\overline{\pi_1} \;\; \overline{\pi_2}\,]$, we have the following system of linear equations i.e.

$$\overline{\pi_1}\,\widetilde{A_{11}} = \bar{0} \;\; i.e.\; boundary\, system\; of\; two\; equations$$

$$\overline{\pi_1}\,\widetilde{A_{12}} + \overline{\pi_2}\,A_{22} = \bar{0}\,.Thus, we\; have\; \overline{\pi_2} = -\overline{\pi_1}\,\widetilde{A_{12}}\,A_{22}^{-1}\,.$$

The boundary system of linear equations leads to a single linear equation in 2 variables i.e. denoting $\overline{\pi_1} = [\pi_1^1 \quad \pi_1^2]$, we have a linear equation of the form

$$\pi_1^1\alpha + \quad \pi_1^2\,\beta = 0.\; Thus,\; \pi_1^2 = -\pi_1^1\,\frac{\alpha}{\beta}.$$

Further, since $\overline{\pi_2}$ can be expressed in terms of $\overline{\pi_1}$, we utilize the normalizing equation $(\overline{\pi_1} + \overline{\pi_2})\;\bar{e} = 1,\; where\; \bar{e}$ is a column vector of ones,

to determine π_1^1 and hence all the other equilibrium probabilities.

Remark 1:

We realize that the inverse of a 2 x 2 matrix can be computed by inspection and essentially only requires computation of the determinant. For instance, let us consider a 2 x 2 matrix B i.e. $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$. $It\; is\; well\; known\; that\; B^{-1} = \frac{1}{\Delta}\begin{bmatrix} b_{22} & -b_{12} \\ -b_{21} & b_{11} \end{bmatrix}$, $where\; \Delta = determinant(B) = \quad (\,b_{11}b_{22} - b_{12}b_{21}\,)$.

COMPUTATIONAL COMPLEXITY: Now, we determine the computational complexity of our algorithm:

- $\pi_1^2\; computation\; requires\; ONE\; division\,.$

- $\overline{\pi_2}$ *computation requires inversion of* $2\,x\,2$ *matrix* A_{22} *and one matrix multiplication* (*of* $2\,x\,2$ *matrices* $\widetilde{A_{12}}$ *and* A_{22}).

- *Also, the normalizing equation to determine* π_1^1 *requires* 3 *additions and ONE division.*

- *Finally, to determine* $\overline{\pi_1}, \overline{\pi_2}$ *in terms of* $\pi_{1,}^1$ *we require* 3 *multiplications.*

Now, we generalize the above algorithm for the case, where the number of states, *N = 2 m where m > 1.* In such case, the generator matrix is of the following form:

$$Q = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \dots & A_{1m} \\ A_{21} & A_{22} & A_{23} & \dots & A_{2m} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ A_{m1} & A_{m2} & A_{m3} & \dots & A_{mm} \end{bmatrix} \quad where\ A'_{ij}s\ are\ 2\ x\ 2\ matrices.$$

From literature, it is well known that in the case of positive recurrent CTMC,

$\{ A_{ii} \colon 1 \leq i \leq m \}$ are all non-singular matrices. Hence, as in the m=1 case, by means of elementary column operations on the generator matrix Q, we arrive at the following block upper triangular matrix $\tilde{Q}$.

$$\tilde{Q} = \begin{bmatrix} \widetilde{A_{11}} & \widetilde{A_{12}} & \widetilde{A_{13}} & \cdots & \widetilde{A_{1\,m-1}} & \widetilde{A_{1m}} \\ 0 & \widetilde{A_{22}} & \widetilde{A_{23}} & \cdots & \widetilde{A_{2\,m-1}} & \widetilde{A_{2m}} \\ \vdots & \vdots & \vdots & & \vdots & \\ 0 & 0 & 0 & \cdots & \widetilde{A_{m-1\ m-1}} & \widetilde{A_{m-1\,m}} \\ 0 & 0 & 0 \dots . & 0 & & \widetilde{A_{mm}} \end{bmatrix}.$$

Thus, the equilibrium probability vector $\bar{\pi}$ satisfies the following linear system of equations:

$$\overline{\pi_1}\,\widetilde{A_{11}} \equiv \bar{0}$$

$$\overline{\pi_1}\,\widetilde{A_{12}} + \overline{\pi_2}\,\widetilde{A_{22}} \equiv \bar{0}$$

$$\vdots$$

$$\overline{\pi_1}\,\widetilde{A_{1m-1}} + \overline{\pi_2}\,\widetilde{A_{2\,m-1}} + \cdots . \overline{\pi_{m-1}}\ \widetilde{A_{m-1\,m-1}} \equiv \bar{0}$$

$$\overline{\pi_1}\,\widetilde{A_{1m}} + \overline{\pi_2}\,\widetilde{A_{2m}} + \cdots + \overline{\pi_m}\ A_{mm} \equiv \bar{0}\,.$$

The above system of linear equations is recursively solved to compute the equilibrium probability vector i.e.

$$\overline{\pi_1}\,\widetilde{A_{11}} \equiv \bar{0}$$

$$\overline{\pi_2} = -\overline{\pi_1}\,\widetilde{A_{12}}\,\widetilde{A_{22}}^{-1}.$$

$$\vdots$$

$$\overline{\pi_J} = -\overline{\pi_1}\,\widetilde{A_{1J}}\,\widetilde{A_{JJ}}^{-1} - \overline{\pi_2}\,\widetilde{A_{2J}}\,\widetilde{A_{JJ}}^{-1} - \cdots\cdots - \overline{\pi_{J-1}}\,\widetilde{A_{J-1\,J}}\,\widetilde{A_{JJ}}^{-1}$$

$$\overline{\pi_m} = -\overline{\pi_1}\,\widetilde{A_{1m}}\,\widetilde{A_{mm}}^{-1} - \overline{\pi_2}\,\widetilde{A_{2m}}\,\widetilde{A_{mm}}^{-1} - \cdots\cdots - \overline{\pi_{m-1}}\,\widetilde{A_{m-1\,m}}\,A_{m\,m}^{-1}\ .$$

Definition: Finite Memory Recursion of order "L" for the equilibrium probability vector is of the following form

$$\bar{\boldsymbol{\pi}}\,(\boldsymbol{L}+\mathbf{1}) = \bar{\pi}\,(1)\,W_1 + \bar{\pi}\,(2)\,W_2 + \cdots + \bar{\pi}\,(L)\,W_L\ ,\qquad where\ W_i's\ \ are\ recursion\ matrices.$$

We call such a recursion as *"forward finite memory recursion"*.

Remark 2:

Since, $\widetilde{A_{JJ}}'s\ \ are\ all\ non-singular\ 2\,x\ 2\ matrices,\ their\ inverse\ can\ be\ efficiently\ computed.$

We now compute the "computational complexity" of the algorithm to Determine equilibrium probabilities in most general case.

(I) 2 x 2 Matrix inversions: We require (m-1) inversions of 2 x 2 matrices. Each 2x2 matrix inversion requires (a) Determinant computation i.e. 2 multiplications and one subtraction (b) Division of elements by determinant i.e. 4 divisions

(II) 2 x 2 Matrix Multiplications: In the above system of linear equations, we require

1 + 2 + 3 +.....+ (m-1) = $\frac{m\,(m-1)}{2}$ multiplications of two 2 x 2 matrices. Each such multiplication (of two 2 x 2 matrices) requires atmost 8 multiplications and 4 additions.

(III) Vector Matrix Multiplications: We require

$$1 + 2 + 3 + \ldots\ldots + (m-1) = \frac{m\,(m-1)}{2}$$

multiplications of 1 x 2 and 2 x 2 matrices (i.e. multiplication of row vector and 2 x 2 matrix). To reduce the complexity, we can utilize Strassen's multiplication algorithm.

(IV) Normalization: It requires (2 m – 1) additions and ONE division to $determine\ \ \pi_1^1$.

(V) Finally, we require (2 m -1) multiplications to determine the equilibrium probabilities using π_1^1 .

Remark 3:

Suppose, the number of states of the CTMC is an ODD number. Then, we consider the 3 x 3 boundary system of linear equations (i.e. the initial probability vector is of dimension 3) and utilize the above idea to compute the equilibrium probability vector efficiently.

Remark 4:

It should be noted that using similar idea (as discussed above), by means of elementary column operations, the generator matrix can be converted into a block lower triangular matrix. In this case, the boundary system of linear equations is at the trailing boundary. By solving for the last probability vector, we recursively compute the equilibrium probabilities. We call such a recursion as the *"backward finite memory recursion*".

3. Efficient Computation of Transient Probability Mass Function of Continuous Time Markov Chains:

It is well known that the transient probability mass function (PMF) of a homogeneous CTMC (generator matrix doesnot depend on time t, unlike non-homogeneous CTMC) satisfies the following vector matrix differential equation i.e.

$$\frac{d}{dt}\ \overline{\pi}\,(t) = \overline{\pi}\,(t)\ Q\ .$$

Taking Laplace-Transform on both sides, we have that

$$s\,\overline{\pi}\,(s) - \overline{\pi}\,(0) = \overline{\pi}\,(s)\ Q\ .$$

Equivalently, we have the following expression for $\overline{\pi}\,(s)$:

$$\overline{\pi}\,(s)\ = -\,\overline{\pi}\,(0)\,[\,Q - s\,I\,]^{-1}.$$

These constitute a "structured" system of linear equations. Our goal is to efficiently solve such system of equations for $\overline{\pi}\,(s)$ and compute the inverse Laplace Transform of $\overline{\pi}\,(s)$ to arrive at the time dependent (transient) PMF of CTMC.

It is well known that in the case of positive recurrent (recurrent non-null) CTMC, in the Region of Convergence (ROC) (of Laplace Transform) the sub-matrices on the diagonal of (Q – s I) (as considered in the equilibrium case) are strictly diagonally dominant and hence are all non-singular. Thus, as in the case of computation of equilibrium PMF, $\overline{\pi}\,(s)$ can be determined efficiently for the values of 's' lying in the Region of Convergence. Detailed Duplication of equations is avoided for brevity.

4. **Lower Bound on Computation of Equilibrium / Transient Probability Mass Function:**

From the above discussion, it is clear that the computation of transient/equilibrium Probability Mass Function (PMF) reduces to solving structured system of linear equations. We exploit the fact that inversion of 2 x 2 matrices (lying on the diagonal of the generator matrix of CTMC) can be carried out in a computationally efficient manner. Using that idea, based on Finite Memory Recursions for the equilibrium/ transient PMF, such probabilities are determined with minimum number of arithmetic operations.

Remark 5:

If the generator matrix of a CTMC is arbitrary (without any specific structure as in the case of say QBD process), the above algorithm MOSTLY achieves the lower bound on computational complexity (i.e. number of arithmetical operations are minimum possible in number). In computational linear algebra, there are well known algorithms to achieve at the LOWER BOUND on solving an arbitrary linear system of equations. We borrow those ideas alongwith the approach proposed in this research paper to efficiently compute the equilibrium/transient PMF of an arbitrary CTMC.

We now provide some numerical results for transient probability mass function of an arbitrary CTMC.

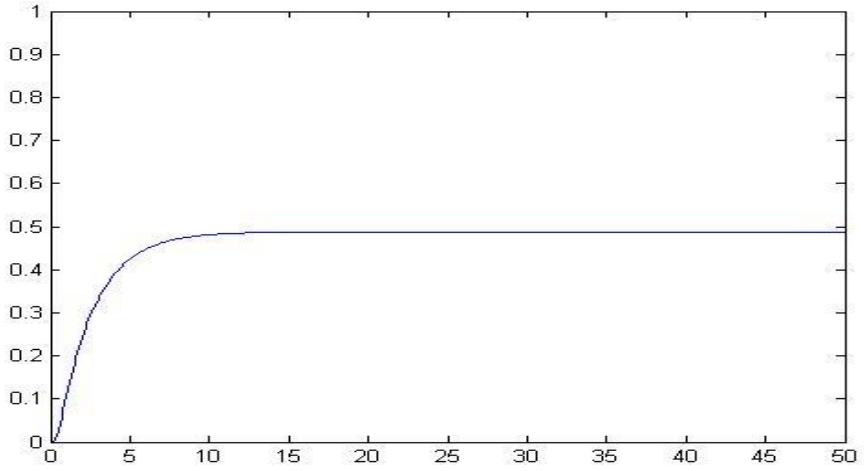


Figure 1. Transient Probability of CTMC being in State 10, as a function of time, given that it starts out in state '1'

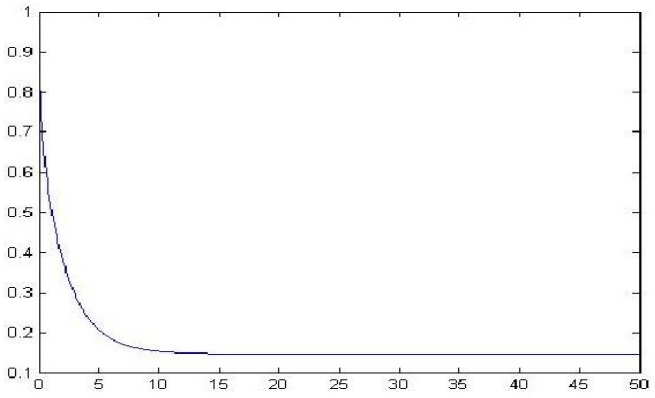
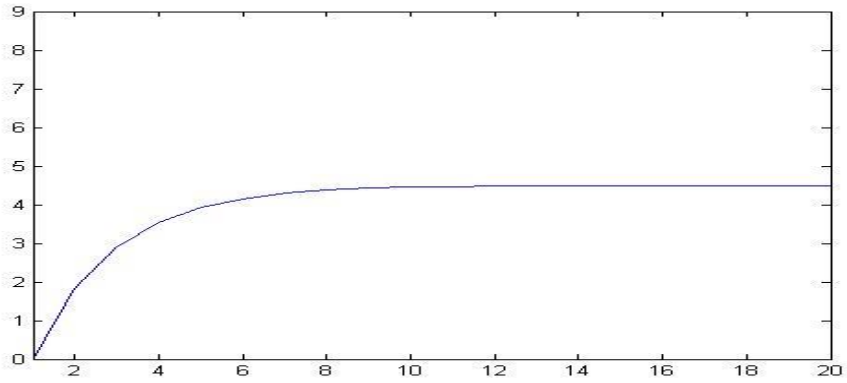
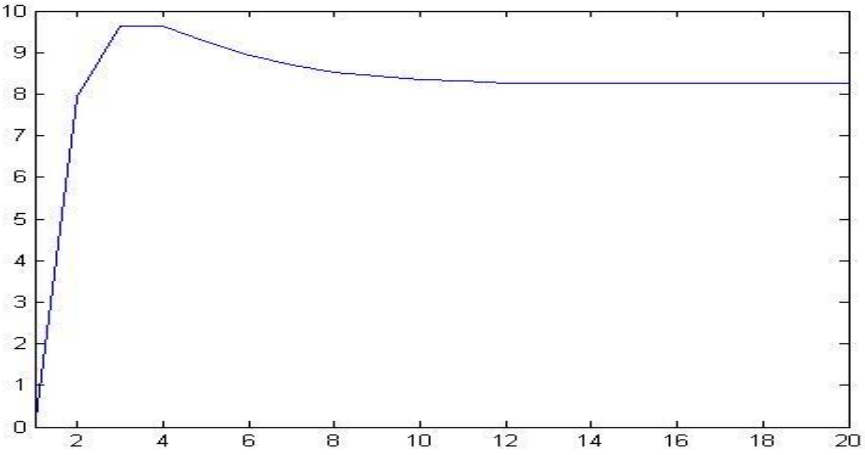


Figure 2: Transient Probability of the CTMC being in state '1' as a function of time, given that the CTMC starts out in state '1'.

- We can plot the expected value of the transient probability distribution and variance of the transient probability distribution (as a function of time) , as follows:



- Variance of the transient probability distribution can also be obtained (as a function of time):



5. Conclusions:

In this research paper, efficient algorithms for computing the equilibrium and transient probability distribution of an arbitrary finite state space Continuous Time Markov Chain are discussed. The algorithms effectively solve a structured system of linear equations efficiently. The algorithms potentially achieve the lower bound on computational complexity.

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