Finite Hilbert-Style Axiomatizations of Disjunctive and Implicative Finitely-Valued Logics with Equality Determinant

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ABSTRACT
Here, we develop a unversal method of [effective] constructing a [finite] Hilbert-style axiomatization of the logic of a given finite disjunctive/implicative matrix with equality determinant [and finitely many connectives] (in particular, any/ implicative four-valued expansion of Belnap's four-valued logic /[as well as any Łukasiewicz finitely-valued logic]). As a by-product, we also prove that the poset of all disjunctive/axiomatic extensions of the logic is dual to the finite distributive lattice of all relatively-hereditary subsets of the set of all consistent submatrices of the matrix [to be found effectively together with their finite relative axiomatizations and both sound and complete matrix semantics].

KEYWORDS
Logic; calculus; sequent; matrix.

1. Introduction

Though various universal approaches to (mainly, *many-place*) *sequent* axiomatizations of finitely-valued logics (cf., *e.g.*, Pynko (2014) as well as both its and its references' bibliographies) have being extensively developed, the problem of their standard (viz., Hilbert-style) axiomatizations (especially, on a generic level) has deserved much less emphasis.

On the other hand, the general study Pynko (2004) has suggested a universal method of [effective] constructing a multi-conclusion *two-side* (as opposed to the above approaches) sequent calculus with structural rules and Cut Elimination Property for a given finite matrix with equality determinant [and finitely many connectives]. In this paper, providing the matrix involved is disjunctive/implicative, we advance the mentioned study by [effective] transforming any [finite] *sequential table for* the matrix (viz., a collection of context-free schemas of uniquely-chosen introduction rules for the matrix and all compound non-nullary connectives not belonging to the equality determinant) and minimal sequent axioms with uniquely-ordered disjoint sides without duplicates consisting of solely either elements of the equality determinant or their values on nullary connectives true in the matrix, actually giving a Gentzen-style axiomatization of the logic of the matrix in Pynko (2004), to a [finite] Hilbert-style

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axiomatization of the logic. As a by-product of this advanced elaboration, we also prove that the poset of all disjunctive/axiomatic extensions of the logic is dual to the finite distributive lattice of all relatively-hereditary subsets of the set of all consistent submatrices of the matrix [to be found effectively together with their finite relative axiomatizations and both sound and complete matrix semantics].

Our general approach, first of all, covers, aside from respective fragments of the classical logic, two especially representative infinite classes of finitely-valued logics: both four-valued expansions of Belnap's useful four-valued logic Belnap (1977), which were started to be studied in Pynko (1999) on an advanced level, and Łukasiewicz finitely-valued logics Łukasiewicz (1920). In addition, it covers miscellaneous three-valued para-consistent/-complete logics. Although most interesting of these are axiomatic/disjunctive extensions of appropriate four-valued expansions of Belnap's four-valued logic, there are certain interesting exceptions (like HZ Hałkowska and Zajac (1988)) deserving a particular emphasis, for which *finite* Hilbert-style axiomatizations have not been found yet.

The rest of the paper is as follows. We entirely follow the standard conventions (as for Hilbert-style calculi) as well as those adopted in both Pynko (1999) and Pynko (2004) — as to sequent calculi. Section 2 is a concise summary of mainly those basic issues underlying the paper, which have proved beyond the scopes of the mentioned papers, those presented therein being normally (though not entirely) briefly summarized as well for the exposition to be properly self-contained. In Section 3 we present a uniform formalism for covering both Hilbert- and Gentzen-style calculi without repeating practically same issues concerning calculi of both kinds, and recall some key results concerning disjunctive and implicative logics (mainly belonging to a logical folklore) and sequent calculi with structural rules going back to Pynko (1999). Then, Section 4 is a preliminary study of minimal disjunctive Hilbert- as well as Gentzen-style (both multi- and single-conclusion) calculi to be used further. Section 5 then contains the main generic results of the paper. Finally, in Section 6 we apply it to disjunctive and implicative positive fragments of the classical logic (mainly, with improving Dyrda and Prucnal (1980)), to Łukasiewicz finitely-valued logics and to both four-valued expansions of Belnap's four-valued logic and their disjunctive extensions as well as to the three-valued logic HZ Hałkowska and Zajac (1988), applications to which look especially acute, because of the infiniteness of its Hilbert-style axiomatization originally found in Zbrzezny (1990). Finally, Section 7 is a brief summary of principal *definitive* contributions of the paper.

2. Basic issues

Notations like img, dom, ker, hom, π_i, R^{-1} and $Q \circ R$ as well as related notions are supposed to be clear.

2.1. Set-theoretical background

We follow the standard set-theoretical convention, according to which natural numbers (including 0) are treated as finite ordinals (viz., sets of lesser natural numbers), the ordinal of all them being denoted by ω (cf., e.g., Mendelson (1979)). The proper class of all ordinals is denoted by ∞. Likewise, functions are viewed as binary relations. In addition, singletons are often identified with their unique elements, unless any confusion is possible.

Given a set S, the set of all subsets of S [of cardinality $\in K \subseteq \infty$] is denoted by $\wp_{[K]}(C)$. A subset of C is said to be *proper*, whenever it is distinct from S. An *enumeration of* S is any bijection from $|S|$ onto S. As usual, given any equivalence relation θ on S, by ν_θ we denote the function with domain S defined by $\nu_\theta(a) \triangleq \theta[\{a\}]$, for all $a \in S$, in which case $\ker \nu_\theta = \theta$, whereas we set $(T/\theta) \triangleq \nu_\theta[T]$, for every $T \subseteq S$. Next, any S-tuple (viz., a function with domain S) is often written in the sequence form $\bar{t}$, its s-th component (viz., the value under argument s) $\pi_s(\bar{t})$, where $s \in S$, being written as t_s, in that case. As usual, given two more sets A and B, any relation between them is identified with the equally-denoted relation between A^S and B^S defined point-wise. Further, elements of $S^* \triangleq (S^0 \cup S^+)$, where $S^+ \triangleq (\bigcup_{i \in (\omega \setminus 1)} S^i)$, are identified with ordinary finite tuples/[comma separated] sequences [in which case, as usual, semicolon instead of comma is sometimes used as sets elements separator to avoid any confusion], the binary concatenation operation on S^* being denoted by $*$, as usual. Then, any $\diamond : (S \times S) \to S$ determines the equally-denoted mapping $\diamond : S^+ \to S$ as follows: by induction on the length (viz., domain) l of any $\bar{a} \in S^+$, put:

$$(\diamond \bar{a}) \triangleq \begin{cases} a_0 & \text{if } l = 1, \\ (\diamond(\bar{a} \upharpoonright (l-1))) \diamond a_{l-1} & \text{otherwise.} \end{cases}$$

Likewise, given a one more set T, any $\diamond : (S \times T) \to T$ determines the equally-denoted mapping $\diamond : (S^* \times T) \to T$ as follows: by induction on the length (viz., domain) l of any $\bar{a} \in S^*$, for all $b \in T$, put:

$$(\bar{a} \diamond b) \triangleq \begin{cases} b & \text{if } l = 0, \\ a_0 \diamond (((\bar{a} \upharpoonright (l \setminus 1)) \circ ((+1) \upharpoonright (l-1))) \diamond b) & \text{otherwise.} \end{cases}$$

Given any $R \subseteq S^2$, put $R^1 \triangleq R$ and $R^0 \triangleq \Delta_S \triangleq \{\langle s, s \rangle \mid s \in S\}$, functions of the latter kind being said to be *diagonal/identity.*

A *[dual] Galois retraction between* posets $\langle P, \leqq \rangle$ and $\langle Q, \lesssim \rangle$ is any couple $\langle f, g \rangle$ of anti-monotonic [resp., monotonic] mappings $f : P \to Q$ and $g : Q \to P$ such that $(g \circ f) = \Delta_P$ and $(f \circ g) \subseteq \lesssim^{[-1]}$, in which case case the former poset is said to be a *[dual] Galois retract of* the latter, while f is a dual embedding [resp., an embedding] of the former into the latter. (Galois retractions are exactly Galois connections with injective/surjective left/right component; cf. Pynko (2000). Moreover, dual Galois retractions between $\langle P, \leqq \rangle$ and $\langle Q, \lesssim \rangle$ are exactly Galois retractions between $\langle P, \leqq \rangle$ and $\langle Q, \lesssim^{-1} \rangle$.)

Let A be a set. It is said to be a(n) *(inclusion) anti-chain*, whenever $\max_\subseteq(A) = A$.

A $U \subseteq \wp(A)$ is said to be *upward-directed*, provided, for every $S \in \wp_\omega(U)$, there is some $T \in U$ such that $(\bigcup S) \subseteq T$. A *closure system over* A is any $\mathcal{C} \subseteq \wp(A)$ such that, for every $S \subseteq \mathcal{C}$, it holds that $(A \cap \bigcap S) \in \mathcal{C}$. An *operator over* A is any unary operation O on $\wp(A)$. This is said to be *(monotonic) [idempotent]* $\{$*transitive*$\}$ $\langle$*inductive/finitary/compact*$\rangle$, provided, for all $(B,)D \in \wp(A)$ $\langle$resp., any upward-directed $U \subseteq \wp(A)\rangle$, it holds that $(O(B))[D]\{O(O(D)\} \subseteq O(D)\langle O(\bigcup U) \subseteq \bigcup O[U]\rangle$. A *closure operator over* A is any monotonic idempotent transitive operator over A.

A *[minimal] covering of* A is any $C \subseteq \wp(A)$ such that $A = (\bigcup C)$ [and no proper subset of C is a covering of A], in which case, providing A is finite, $\max(C) \subseteq C$ is a covering of C, and so any [minimal] covering of a finite set includes a minimal one [resp., is an anti-chain]. A *partition of* A is any covering of A with pair-wise disjoint non-empty elements, in which case it is a minimal covering of A. (Clearly, in case $A \neq \varnothing$, C contains a non-empty set, and so is non-empty and, being an anti-chain, does not contain $\varnothing$. In particular, if $|A| \in 3$ and C is an anti-chain, then C is either $\{A\}$ or $\{\{a\} \mid a \in A\}$, both — but the former not being minimal, if $A = \varnothing$ — being partitions of A, so a covering of a no-more-than-two-element set is minimal iff it is a partition of the set. On the other hand, in case A has three distinct elements a, b and c, both coverings $\{A \setminus \{a\}, A \setminus \{b\}\} \subsetneq \{A \setminus \{a\}, A \setminus \{b\}, A \setminus \{c\}\}$ of A are anti-chains, the former being minimal, but is not a partition of A.) Providing A is finite, a *minimal covering enumeration scheme* (*MCES*, for short) *for* A is any set $\mathcal{M}$ of injective elements of $\wp(A)^*$ such that the function $e_{\mathcal{M}} : \mathcal{M} \to \wp(\wp(A)), f \mapsto (\operatorname{img} f)$ is injective with $\operatorname{img} e_{\mathcal{M}}$ being the set of all minimal coverings of A. (Clearly, if $|A| = 0/1$, then $\varnothing/\{\langle 0, A\rangle\}$ is the only MCES for A. Likewise, if $|A| = 2$, then any MCES for A is of the form $\{\langle 0, A\rangle\} \cup \{\langle i, \{a_i\}\rangle \mid i \in (\operatorname{dom} \bar{a})\}$, where $\bar{a}$ is an enumeration of A.)

2.1.1. Disjunctivity versus multiplicativity

Fix any $\delta : A^2 \to A$. Given any $X, Y \subseteq A$, set $\delta(X, Y) \triangleq \delta[X \times Y]$. Then, a closure operator C over A is said to be *[K-]δ-multiplicative* [where $K \subseteq \infty$] provided

$$\delta(C(X \cup Y), a) \subseteq C(X \cup \delta(Y, a)), \tag{2.1}$$

for all $(X \cup \{a\}) \subseteq A$ and all $Y \in \wp_{[K]}(A)$.[1] Next, C is said to be *δ-disjunctive*, provided, for all $a, b \in A$ and every $X \subseteq A$, it holds that

$$C(X \cup \{\delta(a, b)\}) = (C(X \cup \{a\}) \cap C(X \cup \{b\})), \tag{2.2}$$

in which case the following clearly hold, by (2.2) with $X = \varnothing$:

$$\begin{aligned}
\delta(a, b) &\in C(a), && (2.3)\\
\delta(a, b) &\in C(b), && (2.4)\\
a &\in C(\delta(a, a)), && (2.5)\\
\delta(b, a) &\in C(\delta(a, b)), && (2.6)\\
C(\delta(\delta(a, b), c)) &= C(\delta(a, \delta(b, c))), && (2.7)
\end{aligned}$$

for all $a, b, c \in A$.

Lemma 2.1. *Let C be a [finitary] closure operator over A. Then,* (i)⇔ (ii)⇔(iii)⇐ [⇔](iv)*, where:*

(i) *C is δ-disjunctive;*
(ii) (2.3)*,* (2.5) *and* (2.6) *hold and C is singularly-δ-multiplicative;*
(iii) (2.3)*,* (2.5) *and* (2.6) *hold and C is finitely-δ-multiplicative;*
(iv) (2.3)*,* (2.5) *and* (2.6) *hold and C is δ-multiplicative.*

Proof. First, (ii/iii) is a particular case of (iii/iv), respectively. [Next, (iii)⇒(iv) is by C's being finitary.]

Further, assume (i) holds. Consider any $(X \cup \{a, b\}) \subseteq A$ and any $c \in C(X \cup \{b\})$, in which case $\delta(c, a) \in C(X \cup \{b\})$, by (2.3). Moreover, by (2.4), we also have $\delta(c, a) \in$

[1] In this connection, "finitely-/singularly-" means "ω-/{1}-", respectively.

$C(X\cup\{a\})$. Thus, by (2.2), we get $\delta(c,a)\in(C(X\cup\{b\})\cap C(X\cup\{a\})=C(X\cup\{\delta(b,a)\})$. In this way, (ii) holds.

Next, assume (ii) holds. In that case, both (2.3) and so, by (2.6), (2.4) hold, and so does the inclusion from left to right in (2.2). Conversely, consider any $c\in(C(X\cup\{b\})\cap C(X\cup\{a\}))$, where $(X\cup\{a,b\})\subseteq A$. Then, by (2.6) and (2.1) with $Y=\{a\}$ and b instead of a, we have $\delta(b,c)\in C(X\cup\{\delta(a,b)\})$. Likewise, by (2.5) and (2.1) with $Y=\{b\}$ and c instead of a, we have $c\in C(X\cup\{\delta(b,c)\})$. Therefore, we eventually get $c\in C(X\cup\{\delta(a,b)\})$. Thus, (i) holds.

Finally, assume (i) holds. By induction on any $n\in\omega$, let us show that C is n-δ-multiplicative. For consider any $(X\cup\{a\})\subseteq A$, any $Y\in\wp_n(A)$, in which case $n\neq 0$, and any $b\in C(X\cup Y)$. In case $Y=\varnothing$, (2.1) is by (2.3). Otherwise, take any $c\in Y$, in which case $Y'\triangleq(Y\setminus\{c\})\in\wp_{n-1}(A)$, and put $X'\triangleq(X\cup\{c\})\subseteq A$, in which case $(X'\cup Y')=(X\cup Y)$, and so $b\in C(X'\cup Y')$. Hence, by induction hypothesis, we get $\delta(b,a)\in C(X'\cup\delta(Y',a))=C(\{c\}\cup(X\cup\delta(Y',a)))$. Moreover, by (2.4), we have $\delta(b,a)\in C(\{a\}\cup(X\cup\delta(Y',a)))$. Therefore, as $Y=(Y'\cup\{c\})$, by (2.2), we eventually get $\delta(b,a)\in C(\{\delta(c,a)\}\cup(X\cup\delta(Y',a)))=C(X\cup\delta(Y,a))$. Thus, as $(\bigcup\omega)=\omega$, we conclude that C is finitely-δ-multiplicative, and so (iii) holds, as required. □

2.2. Algebraic background

Unless otherwise specified, all along the paper, we deal with a fixed but arbitrary signature Σ of *primary (propositional) connectives* of finite arity to be treated as operation (viz., function) symbols. Given any $\alpha\in\wp_{\infty\setminus 1}(\omega)$, $\mathfrak{Fm}^{\alpha}_{\Sigma}$ denotes the absolutely-free Σ-algebra freely-generated by the set $V_\alpha\triangleq\{x_i\mid i\in\alpha\}$ of *(propositional) variables*, its endomorphisms/elements of its carrier $\mathrm{Fm}^{\alpha}_{\Sigma}$ being called *(propositional) Σ-substitutions/formulas*, in case $\alpha=\omega$. As usual, a *secondary (propositional) connective of Σ of arity* $n\in\omega$ is any element of $\mathrm{Fm}^{\max(n,1)}_{\Sigma}$, any primary $F\in\Sigma$ of arity $n\in\omega$ being naturally identified with the secondary one $F(x_i)_{i\in n}$. The function $\mathrm{Var}:\mathrm{Fm}^{\omega}_{\Sigma}\to\wp_\omega(V_\omega)$, assigning to every $\varphi\in\mathrm{Fm}^{\omega}_{\Sigma}$ the finite set of all variables *actually* occurring in φ, is defined in the standard recursive way by induction on construction of φ. For any $\Pi\subseteq\mathrm{Fm}^{\omega}_{\Sigma}$, set $\mathrm{Fm}^{\alpha}_{\Pi}\triangleq(\bigcap\{V_\alpha\subseteq S\subseteq\mathrm{Fm}^{\alpha}_{\Sigma}\mid\forall\sigma\in\hom(\mathfrak{Fm}^{\omega}_{\Sigma},\mathfrak{Fm}^{\omega}_{\Sigma}):(\sigma[V_\omega]\subseteq S)\Rightarrow(\sigma[\Pi]\subseteq S)\})\subseteq\mathrm{Fm}^{\alpha}_{\Sigma}$.

As usual, *(logical) Σ-matrices* (cf. Loś and Suszko (1958)) are treated as first-order model structures (viz., algebraic systems; cf. Mal'cev (1965)) of the first-order signature $\Sigma\cup\{D\}$ with unary *truth* predicate D. In general, [Σ-matrices are denoted by Calligraphic letters (possibly, with indices), their *underlying*] algebras [viz., their Σ-reducts] being denoted by [corresponding] Fraktur letters (possibly, with [same] indices [if any]), their carriers being denoted by corresponding Italic letters (with same indices, if any). Any Σ-matrix $\mathcal{A}$ is traditionally identified with the couple $\langle\mathfrak{A},D^{\mathcal{A}}\rangle$. This is said to be *[in]cosistent/truth[-non]-empty*, if $((A\setminus D^{\mathcal{A}})/D^{\mathcal{A}})\neq[=]\varnothing$. Given a class M of Σ-matrices, $\mathbf{S}_{[*]}(\mathsf{M})$ denotes the class of all [consistent] submatrices of members of M, M being said to be *[consistently] hereditary*, whenever it includes $\mathbf{S}_{[*]}(\mathsf{M})$. In general, $\mathbf{S}_{[*]}(\mathsf{M})$ is the least [consistently] hereditary class including M and called the one *generated by* M. As usual, the class of all models of any first-order theory $\mathcal{T}$ of the first-order signature $\Sigma\cup\{D\}$ is denoted by $\mathrm{Mod}(\mathcal{T})$, $\mathcal{T}$ being said to *axiomatize* $\mathrm{Mod}(\mathcal{T})[\cap\mathsf{M}$ *relatively to* $\mathsf{M}]$.

2.2.1. Equality determinants for matrices

According to Pynko (2004), an *equality determinant for* a Σ-matrix $\mathcal{A}$ is any $\Upsilon \subseteq \mathrm{Fm}^1_\Sigma$ such that any $a, b \in A$ are equal, whenever, for all $\upsilon \in \Upsilon$, $\upsilon^{\mathfrak{A}}(a) \in D^{\mathcal{A}}$ iff $\upsilon^{\mathfrak{A}}(b) \in D^{\mathcal{A}}$, in which case $\Upsilon \cap \mathrm{Var}^{-1}[\{V_1\}]$ is so.

3. Abstract propositional languages and calculi

A(n) *(abstract)* Σ-*[propositional]language* is any triple of the form $L = \langle \mathrm{Fm}_L, \Im_L, \mathrm{Var}_L\rangle$, where Fm_L is a set, whose elements are called *L-formulas*, while $\Im_L : \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{Fm}^\omega_\Sigma) \to (\mathrm{Fm}_L)^{\mathrm{Fm}_L}$, preserving compositions and diagonality, any Σ-substitution σ being naturally identified with $\Im_L(\sigma)$, unless any confusion is possible, whereas $\mathrm{Var}_L : \mathrm{Fm}_L \to \wp_\omega(V_\omega)$, the language subscript being in general normally omitted, unless any confusion is possible, such that, for every $\Phi \in \mathrm{Fm}_L$ and any Σ-substitutions σ and ς such that $(\sigma{\restriction}\mathrm{Var}_L(\Phi)) = (\varsigma{\restriction}\mathrm{Var}_L(\Phi))$, it holds that $\sigma(\Phi) = \varsigma(\Phi)$. Given any $V \subseteq V_\omega$, set $\mathrm{Fm}_L(V) \triangleq (\mathrm{Fm}_L \cap \mathrm{Var}^{-1}_L[\wp_\omega(V)])$.

Then, elements/subsets of $\mathrm{Ru}_L \triangleq (\wp_\omega(\mathrm{Fm}_L) \times \mathrm{Fm}_L)$ are referred to as *L-rules/calculi*, any L-rule $\mathcal{R} = \langle\Gamma, \Phi\rangle$ being normally written in either conventional displayed $\frac{\Gamma}{\Phi}$ or non-displayed $\Gamma|\Phi$ form, Φ/any element of Γ being called the/a *conclusion/premise of* $\mathcal{R}$, rules of the form $\Phi|\Psi$, where $\Psi \in \Gamma$, being said to be *inverse to* $\mathcal{R}$. As usual, L-rules without premises are called *L-axioms* and are identified with their conclusions, calculi consisting of merely axioms being said to be *axiomatic*. In general, any function f with domain Fm_L (including Σ-substitutions) but Var_L determines the equally-denoted function with domain Ru_L as follows: for any $\mathcal{R} = \langle\Gamma, \Phi\rangle \in \mathrm{Ru}_L$, we set $f(\mathcal{R}) \triangleq \langle f[\Gamma], f(\Phi)\rangle$, whereas put $\mathrm{Var}_L(\mathcal{R}) \triangleq (\mathrm{Var}_L(\Phi) \cup \bigcup \mathrm{Var}_L[\Gamma]) \in \wp_\omega(V_\omega)$.

Next, an *L-logic* is any closure operator C on Fm_L that is *structural* in the sense that, for every Σ-substitution σ and all $\Gamma \subseteq \mathrm{Fm}_L$, it holds that $\sigma[C(\Gamma)] \subseteq C(\sigma[\Gamma])$. This is said to be *[in]consistent*, if $C(\varnothing) \neq [=]\,\mathrm{Fm}_L$, the only inconsistent L-logic $\wp(\mathrm{Fm}_L) \times \{\mathrm{Fm}_L\}$ being denoted by IC_L, and to *satisfy* an L-rule $\Gamma|\Phi$, whenever $\Phi \in C(\Gamma)$, L-axioms satisfied by C being called its *theorems*. Then, an L-logic C' is said to be a *[proper] extension of* C, provided $C \subseteq C'[\neq C]$, C being referred to as a *[proper] sublogic of* C', respectively. In that case, an L-calculus $\mathcal{C}$ is said to *axiomatize* C' *relatively to* C, provided C' is the least extension of C satisfying each rule in $\mathcal{C}$, extensions of C relatively axiomatized by axiomatic calculi being said to be *axiomatic*. Next, $C_{+0} \triangleq ((C{\restriction}\wp_{\infty\setminus 1}(\mathrm{Fm}_L)) \cup \{\langle\varnothing, \varnothing\rangle\})$, being the greatest sublogic of C without theorems, is called the *theorem-less version of* C. Likewise, the L-logic $C_\exists$, defined by $C_\exists(X) \triangleq (\bigcup C[\wp_\omega(X)])$, for all $X \subseteq \mathrm{Fm}_L$, being the greatest finitary sublogic of C, is called the *finitarization of* C. Finally, $\equiv_C \triangleq \{\langle\Phi, \Psi\rangle \in \mathrm{Fm}^2_L \mid C(\Phi) = C(\Psi)\}$ is an equivalence relation on Fm_L.

Further, an L-formula Φ is said to be *derivable from* $\Gamma \subseteq \mathrm{Fm}_L$ *in* an L-calculus $\mathcal{C}$, if there is a $\mathcal{C}$-*derivation of* Φ *from* Γ, i.e., a proof of Φ (in the standard proof-theoretical sense) by means of axioms in Γ (as *hypotheses*) and rules in the set $\mathrm{SI}_\Sigma(\mathcal{C}) \triangleq \{\sigma(\mathcal{R}) \mid \mathcal{R} \in \mathcal{C}, \sigma \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{Fm}^\omega_\Sigma)\}$ of all *(substitutional)* Σ-*instances of* rules in $\mathcal{C}$. Then, an L-rule is said to be *derivable in* $\mathcal{C}$, if there is a $\mathcal{C}$-*derivation of* it (viz., a $\mathcal{C}$-derivation of its conclusion from the set of its premises). The extension $\mathrm{Cn}_{\mathcal{C}}$ of the diagonal Σ-logic relatively axiomatized by $\mathcal{C}$ is called the *derivability/consequence of* $\mathcal{C}$ and said to be *axiomatized by* $\mathcal{C}$, in which case it is finitary and, for all $(\Gamma \cup \{\Phi\}) \subseteq \mathrm{Fm}_L$, $\Phi \in \mathrm{Cn}_{\mathcal{C}}(\Gamma)$ iff Φ is derivable from Γ in $\mathcal{C}$. (Conversely, any finitary L-logic is axiomatized by the set of all L-rules satisfied in it to be identified with the logic, in which case finitary

L-logics become actually particular cases of L-calculi.) An $S \subseteq \mathrm{Fm}_L$ is said to be $\mathcal{C}$-*closed*, if, for every $(\Gamma|\Phi) \in \mathrm{SI}_\Sigma(\mathcal{C})$, it holds that $(\Gamma \subseteq S) \Rightarrow (\Phi \in S)$, in which case, by induction on the length of $\mathcal{C}$-derivations, it is $\mathrm{Cn}_{\mathcal{C}}$-closed, that is, $S \in (\mathrm{img}\,\mathrm{Cn}_{\mathcal{C}})$, and so, in particular, $\mathrm{Cn}_{\mathcal{C}}(\varnothing) \subseteq S$.

3.1. Hilbert-style calculi

The Σ-language $\mathcal{H}_\Sigma$ with the first component $\mathrm{Fm}^\omega_\Sigma$, the diagonal second component and the third component Var is called the *Hilbert-style/sentential* Σ-*language*, $\mathcal{H}_\Sigma$-rules/-axioms/-calculi/-logics being traditionally referred to as *(Hilbert-style/sentential)* Σ-*rules/-axioms/-calculi/-logics*, respectively (cf. Loś and Suszko (1958)).

From the model-theoretic point of view, any Σ-rule $\Gamma|\phi$ is viewed as (the universal closure of) the first-order basic Horn formula $(\bigwedge \Gamma) \to \phi$ under the standard identification of any Σ-formula ψ with the first-order atomic formula $D(\psi)$ we follow tacitly, Σ-calculi being treated as equality-free universal Horn theories of the first-order signature $\Sigma \cup \{D\}$.

Given any class M of Σ-matrices and any $\alpha \in \wp_{\infty\setminus 1}(\omega)$, we have the closure operator $\mathrm{Cn}^\alpha_{\mathsf{M}}$ over $\mathrm{Fm}^\alpha_\Sigma$, defined by $\mathrm{Cn}^\alpha_{\mathsf{M}}(X) \triangleq (\mathrm{Fm}^\alpha_\Sigma \cap \bigcap\{h^{-1}[D^{\mathcal{A}}] \supseteq X | \mathcal{A} \in \mathsf{M}, h \in \hom(\mathfrak{Fm}^\alpha_\Sigma, \mathfrak{A})\})$, for all $X \subseteq \mathrm{Fm}^\alpha_\Sigma$, in which case:

$$\mathrm{Cn}^\alpha_{\mathsf{M}}(X) = (\mathrm{Fm}^\alpha_\Sigma \cap \mathrm{Cn}^\omega_{\mathsf{M}}(X)), \tag{3.1}$$

because $\hom(\mathfrak{Fm}^\alpha_\Sigma, \mathfrak{A}) = \{h \restriction \mathrm{Fm}^\alpha_\Sigma \,| h \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{A})\}$, for any Σ-algebra $\mathfrak{A}$, as $A \neq \varnothing$, being a[n inconsistent] Σ-logic, whenever $\alpha = \omega$ [and every member of M is inconsistent], and called the one *of/defined by* M, in that case. (Due to Loś and Suszko (1958)/the Compactness Theorem Mal'cev (1965), this is well known to be finitary, whenever both M and all members of it are finite.)

Remark 3.1. Since any Σ-rule [without premises] is [not] true in any truth-empty Σ-matrix, given any class M of Σ-matrices and any non-empty class S of truth-empty Σ-matrices, the logic of $\mathsf{S} \cup \mathsf{M}$ is the theorem-less version of the logic of M. □

A Σ-matrix $\mathcal{A}$ is said to be $\diamond$-*disjunctive/-implicative*, where $\diamond$ is a (possibly, secondary) binary connective of Σ, whenever, for all $a, b \in A$, it holds that $((a \notin / \in D^{\mathcal{A}}) \Rightarrow (b \in D^{\mathcal{A}})) \Leftrightarrow ((a \diamond^{\mathfrak{A}} b) \in D^{\mathcal{A}})/$, in which case it is $\veebar_\diamond$-disjunctive, where $(x_0 \veebar_\diamond x_1) \triangleq ((x_0 \diamond x_1) \diamond x_1)$. Finally, $\mathcal{A}$ is said to be a *model of* a Σ-logic C, whenever C is a sublogic of the logic of $\mathcal{A}$, the class of all them being denoted by $\mathrm{Mod}(C)$, that fits well the model-theoretic conventions adopted above, in case C is finitary.

3.1.1. Disjunctive sentential logics and matrices

Throughout the rest of the paper, unless otherwise specified, $\veebar$ is supposed to be any (possibly, secondary) binary connective of Σ.

Lemma 3.2. *Let* M *be a class of* $\veebar$*-disjunctive* Σ*-matrices. Then, the logic of* M *is* $\veebar$*-multiplicative, and so* $\veebar$*-disjunctive.*

Proof. Consider any $(X \cup Y \cup \{\psi\}) \subseteq \mathrm{Fm}^\omega_\Sigma$, any $\phi \in \mathrm{Cn}_{\mathsf{M}}(X \cup Y)$, any $\mathcal{A} \in \mathsf{M}$ and any $h \in \hom(\mathfrak{Fm}^\omega_\Sigma, \mathfrak{A})$ such that $(h(\phi) \veebar^{\mathfrak{A}} h(\psi)) = h(\phi \veebar \psi) \notin D^{\mathcal{A}}$, in which case $h(\phi) \notin D^{\mathcal{A}} \not\ni h(\psi)$, for $\mathcal{A}$ is $\veebar$-disjunctive, and so $h(\varphi) \notin D^{\mathcal{A}}$, for some $\varphi \in (X \cup Y)$, in which case $h(\varphi \veebar \psi) = (h(\phi) \veebar^{\mathfrak{A}} h(\psi)) \notin D^{\mathcal{A}}$, and so $(\phi \veebar \psi) \in \mathrm{Cn}_{\mathsf{M}}(X \cup (Y \veebar \psi))$. Then, Lemma 2.1(iv)⇒(i) completes the proof, for Cn_{M} satisfies (2.3), (2.5) and (2.6). □

Given a Σ-rule $\Gamma|\phi$ and a Σ-formula ψ, put $((\Gamma|\phi) \veebar \psi) \triangleq ((\Gamma \veebar \psi)|(\phi \veebar \psi))$. (This notation is naturally extended to Σ-calculi member-wise.)

Theorem 3.3. *Let C be a finitary Σ-logic. Then, C is $\veebar$-disjunctive iff* (2.3), (2.5) *and* (2.6) *hold and, for any axiomatization $\mathcal{C}$ of C, every $(\Gamma|\phi) \in \mathrm{SI}_{\Sigma}(\mathcal{C})$ and each $\psi \in \mathrm{Fm}^{\omega}_{\Sigma}$, it holds that $(\phi \veebar \psi) \in C(\Gamma \veebar \psi)$.*

Proof. By Corollary 2.1(i)⇔(iv) and the structurality of C, with using (2.3) and the induction on the length of $\mathcal{C}$-derivations. □

Lemma 3.4. *Let $\mathcal{R} = (\Gamma|\phi)$ be a Σ-rule, C a Σ-logic, $\psi \in \mathrm{Fm}^{\omega}_{\Sigma}$, $\sigma \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{Fm}^{\omega}_{\Sigma})$ and $v \in (V_{\omega} \setminus \mathrm{Var}(\mathcal{R}))$. Suppose* (2.7) *holds and $\mathcal{R} \veebar v$ is satisfied in C. Then, so is $\sigma(\mathcal{R} \veebar v) \veebar \psi$.*

Proof. Let $\varsigma \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{Fm}^{\omega}_{\Sigma})$ extend $(\sigma{\restriction}(V_{\omega} \setminus \{v\})) \cup [v/(\sigma(v) \veebar \psi)]$, in which case $\sigma(\mathcal{R}) = \varsigma(\mathcal{R})$, for $v \notin \mathrm{Var}(\mathcal{R})$. Then, using (2.7) and the structurality of C, we eventually get $(\sigma(\phi \veebar v) \veebar \psi) = ((\sigma(\phi) \veebar \sigma(v)) \veebar \psi) \in C(\sigma(\phi) \veebar (\sigma(v) \veebar \psi)) = C(\varsigma(\phi) \veebar \varsigma(v)) = C(\varsigma(\phi \veebar v)) \subseteq C(\varsigma[\Gamma \veebar v]) = C(\varsigma[\Gamma] \veebar \varsigma(v)) = C(\sigma[\Gamma] \veebar (\sigma(v) \veebar \psi)) = C((\sigma[\Gamma] \veebar \sigma(v)) \veebar \psi) = C(\sigma[\Gamma \veebar v] \veebar \psi)$, as required. □

Let σ_{+1} be the Σ-substitution extending $[x_i/x_{i+1}]_{i \in \omega}$.

Corollary 3.5. *Let C be a finitary $\veebar$-disjunctive logic, $\mathcal{C}$ a Σ-calculus and $\mathcal{A}$ an axiomatic Σ-calculus. Then, the extension C' of C relatively axiomatized by $\mathcal{C}' \triangleq (\mathcal{A} \cup (\sigma_{+1}[\mathcal{C}] \veebar x_0))$ is $\veebar$-disjunctive. In particular, any axiomatic extension of C is $\veebar$-disjunctive.*

Proof. Then, C being finitary, is axiomatized by a Σ-calculus $\mathcal{C}''$, in which case C' is axiomatized by the Σ-calculus $\mathcal{C}'' \cup \mathcal{C}'$, and so is finitary too. Moreover, C', being an extension of C, inherits (2.3), (2.5), (2.6) and (2.7) held for C. Then, we prove the $\veebar$-disjunctivity of C' with applying Theorem 3.3 to both C and C'. For consider any Σ-substitution σ and any $\psi \in \mathrm{Fm}^{\omega}_{\Sigma}$. First, for any $\phi \in \mathcal{A} \subseteq \mathcal{C}'$, by the structurality of C' and (2.3), we have $(\sigma(\phi) \veebar \psi) \in C'(\varnothing)$. Now, consider any $\mathcal{R} \in \mathcal{C}$, in which case $(\sigma_{+1}(\mathcal{R}) \veebar x_0) \in \mathcal{C}'$ is satisfied in C' and $x_0 \in (V_{\omega} \setminus \mathrm{Var}(\sigma_{+1}(\mathcal{R})))$. In this way, Lemma 3.4 with C' and $\sigma_{+1}(\mathcal{R})$ instead of C and $\mathcal{R}$, respectively, completes the argument. □

Proposition 3.6. *Let M be a [finite] class of [finite $\veebar$-disjunctive] Σ-matrices. Then, $\mathbf{S}_*(\mathsf{M})$ has no truth-empty member if[f] the logic of M has a theorem.*

Proof. The "if" part is by Remark 3.1. [Conversely, assume $\mathbf{S}_*(\mathsf{M})$ has no truth-empty member. Let $\overline{\mathcal{A}}$ be any enumeration of M. Consider any $i \in |\mathsf{M}| \in \omega$. Let $\bar{a}$ be any enumeration of $A_i \setminus D^{\mathcal{A}_i}$. Consider any $j \in (\mathrm{dom}\,\bar{a}) \in \omega$. Let $\mathfrak{B}$ be the subalgebra of $\mathcal{A}_i$ generated by $\{a_j\}$. Then, $(\mathcal{A}_i{\restriction}B) \in \mathbf{S}_*(\mathsf{M})$ is truth-non-empty, in which case there is some $\phi_j \in \mathrm{Fm}^{1}_{\Sigma}$ such that $\phi_j^{\mathfrak{A}_i}(a_j) \in D^{\mathcal{A}_i}$, and so $\psi_i \triangleq (\veebar\langle\bar{\phi}, x_0\rangle)$ is true in $\mathcal{A}_i$. In this way, $\veebar\langle\bar{\psi}, x_0\rangle$ is true in M, as required.] □

3.1.2. Implicative sentential logics

Throughout the rest of the paper, unless otherwise specified, $\rhd$ is supposed to be any (possibly, secondary) binary connective of Σ.

A Σ-logic C is said to be $\rhd$-*implicative*, whenever it has *Deduction Theorem* (*DT*, for short) *with respect to* $\rhd$ in the sense that:

$$(\psi \in C(\Gamma \cup \{\phi\})) \Rightarrow ((\phi \rhd \psi) \in C(\Gamma), \tag{3.2}$$

for all $(\Gamma \cup \{\phi, \psi\}) \subseteq \mathrm{Fm}^{\omega}_{\Sigma}$, as well as satisfies both the *Modus Ponens* rule:

$$\frac{x_0 \quad x_0 \rhd x_1}{x_1}, \tag{3.3}$$

and *Peirce Law* axiom (cf. Peirce (1885)):

$$(((x_0 \rhd x_1) \rhd x_0) \rhd x_0). \tag{3.4}$$

As it is well-known, C satisfies the following axioms:

$$x_0 \rhd (x_1 \rhd x_0) \tag{3.5}$$
$$(x_0 \rhd (x_1 \rhd x_2)) \rhd ((x_0 \rhd x_1) \rhd (x_0 \rhd x_2)) \tag{3.6}$$

whenever it has DT with respect to $\rhd$ and satisfies (3.3).

Lemma 3.7. *Any* $\rhd$*-implicative* Σ*-logic is* $\veebar_{\rhd}$*-disjunctive.*

Proof. With using Lemma 2.1(ii)⇒(i). First, (2.3) is by (3.3) and (3.2). Next, (2.5) is by (3.3) and (3.4)$[x_1/x_0]$. Further, by (3.2), (3.3) and (3.4), we have $x_0 \in C(\{x_0 \veebar_{\rhd} x_1, x_1 \rhd x_0\})$, in which case, by (3.2), we get $(x_1 \veebar_{\rhd} x_0) \in C(x_0 \veebar_{\rhd} x_1)$, and so (2.6) holds. Finally, consider any $(\Gamma \cup \{\phi, \psi\}) \subseteq \mathrm{Fm}^{\omega}_{\Sigma}$ and any $\varphi \in C(\Gamma \cup \{\phi\})$, in which case, by (3.2), we have $(\phi \rhd \varphi) \in C(\Gamma)$, and so, by (3.2) and (3.3), we get $\psi \in C(\Gamma \cup \{\phi \veebar_{\rhd} \psi, \varphi \rhd \psi\})$. Hence, by (3.2), we eventually get $(\varphi \veebar_{\rhd} \psi) \in C(\Gamma \cup \{\phi \veebar_{\rhd} \psi\})$. Thus, C is singularly-$\veebar_{\rhd}$-multiplicative, as required. □

By $\mathcal{I}^{[\mathrm{PL}]}_{\rhd}$ we denote the Σ-calculus constituted by (3.3), (3.5) and (3.6) [as well as (3.4)]. Recall the following well-known observation proved by induction on the length of $(\mathcal{I}_{\rhd} \cup \mathcal{A})$-derivations (cf., e.g., Mendelson (1979)):

Lemma 3.8. *Let* $\mathcal{A}$ *be an axiomatic* Σ*-calculus. Then,* $\mathrm{Cn}_{\mathcal{I}_{\rhd} \cup \mathcal{A}}$ *has DT with respect to* $\rhd$*.*

Combining Lemmas 3.7 and 3.8, we eventually get:

Theorem 3.9. *Let* $\mathcal{A}$ *be an axiomatic* Σ*-calculus. Then,* $\mathrm{Cn}_{\mathcal{I}^{\mathrm{PL}}_{\rhd} \cup \mathcal{A}}$ *is* $\rhd$*-implicative, and so* $\veebar_{\rhd}$*-disjunctive.*

Corollary 3.10. *Let* $\mathcal{A} \cup \{\varphi\}$ *be an axiomatic* Σ*-calculus,* $n \in (\omega \setminus 1)$*,* $\bar{\psi} \in (\mathrm{Fm}^{\omega}_{\Sigma})^n$*,* $\bar{\phi} \in (\mathrm{Fm}^{\omega}_{\Sigma})^*$*,* $v \in (V_\omega \setminus (\bigcup \mathrm{Var}[\{\varphi\} \cup ((\mathrm{img}\,\bar{\psi}) \cup (\mathrm{img}\,\bar{\phi}))]))$ *and* $\bar{\zeta} \triangleq \langle \bar{\phi} \rhd (\psi_i \rhd v)\rangle_{i \in n}$*. Then, the following hold:*

(i) *the* Σ*-axiom* $\bar{\phi} \rhd ((\veebar_{\rhd} \bar{\psi}) \rhd \varphi)$ *is derivable in* $\mathcal{I}^{\mathrm{PL}}_{\rhd} \cup \mathcal{A}$ *iff the* Σ*-axioms* $\bar{\phi} \rhd (\psi_i \rhd \varphi)$*, where* $i \in n$*, are so;*

(ii) *the* Σ*-axiom* $\bar{\phi} \rhd (\varphi \rhd (\veebar_{\rhd} \bar{\psi}))$ *is derivable in* $\mathcal{I}^{\mathrm{PL}}_{\rhd} \cup \mathcal{A}$ *iff the* Σ*-axiom* $(\bar{\zeta} \rhd (\bar{\phi} \rhd (\varphi \rhd v))$ *is so.*

Proof. In that case, by Theorem 3.9, $\mathrm{Cn}_{\mathcal{J}^{\mathrm{PL}}_{\rhd}\cup\mathcal{A}}$ is $\rhd$-implicative and $\veebar_{\rhd}$-disjunctive. Then, (2.2) with $X = (\mathrm{img}\,\bar{\phi})$, (3.2), (3.3) and the induction on n immediately yield (i). Next, the "if" part of (i) with v and $\bar{\zeta} * \bar{\phi}$ instead of φ and $\bar{\phi}$, respectively, (3.2) and (3.3) yield the "only if" part of (ii). Finally, applying the substitution $[v/(\veebar_{\rhd}\bar{\psi})]$, the "only if" part of (i) with $\veebar_{\rhd}\bar{\psi}$ instead of φ, (3.3) and (3.5) imply the "if" part of (ii), as required. $\square$

3.2. Gentzen-style calculi

Given any $(\alpha[\cup\beta]) \subseteq \omega$, elements of $\mathrm{Seq}_{\Sigma}^{[\beta\vdash]\alpha} \triangleq \{\langle\Gamma,\Delta\rangle \in ((\mathrm{Fm}_{\Sigma}^{\omega})^*)^2 \mid (\mathrm{dom}\,\Delta) \in \alpha\,[\&\,(\mathrm{dom}\,\Gamma) \in \beta]\}$ are called *α-conclusion [β-premise] Σ-sequents*, "[purely] single/multi" standing for "$(2/\omega)[\setminus 1]$", respectively. Any sequent $\langle\Gamma,\Delta\rangle$ is normally written in the conventional form $\Gamma \vdash \Delta$. This is said to be *injective*, whenever both Γ and Δ are so. Likewise, it is said to be *disjoint*, whenever $((\mathrm{img}\,\Gamma) \cap (\mathrm{img}\,\Delta)) = \varnothing$. For any $\Phi = (\Gamma \vdash \Delta) \in \mathrm{Seq}_{\Sigma}^{[\beta\vdash]\alpha}$, set $\mathrm{Var}(\Phi) \triangleq (\bigcup \mathrm{Var}[\mathrm{img}(\Gamma * \Delta)]) \in \wp_{\omega}(V_{\omega})$ and $\sigma(\Phi) \triangleq ((\sigma\circ\Gamma) \vdash (\sigma\circ\Delta)) \in \mathrm{Seq}_{\Sigma}^{[\beta\vdash]\alpha}$, where σ is a Σ-substitution. In this way, $\mathrm{Seq}_{\Sigma}^{[\beta\vdash]\alpha}$ forms a Σ-language $\mathcal{S}_{\Sigma}^{[\beta\vdash]\alpha}$, called the *$\alpha$-conclusion [$\beta$-premise] Gentzen-style/sequent Σ-language*, $\mathcal{S}_{\Sigma}^{[\beta\vdash]\alpha}$-rules/-axioms/-calculi/logics being referred to as *α-conclusion [β-premise] (Gentzen-style/sequent) Σ-rules/-axioms/-calculi/-logics*, respectively.

The following multi-conclusion sequent $\varnothing$-rules are said to be *structural*:

Reflexivity	$x_0 \vdash x_0$	
Cut	$\dfrac{\Lambda,\Gamma \vdash \Delta, x_0 \quad \Gamma, x_0 \vdash \Delta,\Theta}{\Lambda,\Gamma \vdash \Delta,\Theta}$	
Enlargement	$\dfrac{\Gamma \vdash \Delta}{x_0,\Gamma \vdash \Delta}$	$\dfrac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, x_0}$
Contraction	$\dfrac{x_0,x_0,\Gamma \vdash \Delta}{x_0,\Gamma \vdash \Delta}$	$\dfrac{\Gamma \vdash \Delta, x_0, x_0}{\Gamma \vdash \Delta, x_0}$
Permutation	$\dfrac{\Lambda,x_0,x_1,\Gamma \vdash \Delta}{\Lambda,x_1,x_0,\Gamma \vdash \Delta}$	$\dfrac{\Gamma \vdash \Delta, x_0, x_1, \Theta}{\Gamma \vdash \Delta, x_1, x_0, \Theta}$

where $\Lambda,\Gamma,\Delta,\Theta \in V_{\omega}^*$, Enlargement, Contraction and Permutation being referred to as *basic structural*.

Given two (purely) multi-conclusion [{purely} multi-premise] Σ-sequents $\Phi = (\Gamma \vdash \Delta)$ and $\Psi = (\Lambda \vdash \Theta)$, we have their *sequent disjunction/implication/[diagonal] subsumption*, $\sqsubseteq_{[1]}$ being a quasi-ordering on $\mathrm{Seq}_{\Sigma}^{\omega}$:

$$\begin{array}{rcl}
(\Phi \uplus \Psi) & \triangleq & (\Gamma,\Lambda \vdash \Delta,\Theta) \in \mathrm{Seq}_{\Sigma}^{[(\omega\{\setminus 1\})\vdash](\omega(\setminus 1))} \;/ \\
(\Phi \sqsupset \Psi) & \triangleq & (\{\phi,\Gamma \vdash \Delta \mid \phi \in (\mathrm{img}\,\Theta)\} \\
 & \cup & \{\Gamma \vdash \Delta,\psi \mid \psi \in (\mathrm{img}\,\Lambda)\}) \in \wp_{\omega}(\mathrm{Seq}_{\Sigma}^{[(\omega\{\setminus 1\})\vdash](\omega(\setminus 1))})/ \\
(\Phi \sqsubseteq_{[1]} \Psi) & \stackrel{\mathrm{def}}{\Longleftrightarrow} & \exists\sigma[= \Delta_{\mathrm{Fm}_{\Sigma}^{\omega}}] \in \hom(\mathfrak{Fm}_{\Sigma}^{\omega},\mathfrak{Fm}_{\Sigma}^{\omega}) : \\
 & & (\sigma[\mathrm{img}\,\Gamma] \subseteq (\mathrm{img}\,\Lambda))\&(\sigma[\mathrm{img}\,\Delta] \subseteq (\mathrm{img}\,\Theta)),
\end{array}$$

respectively. Then, given any $X \in \wp_{\langle\omega\rangle}(\mathrm{Seq}_{\Sigma}^{[(\omega\{\setminus 1\})\vdash](\omega(\setminus 1))})$, set $(\Phi \sqsupset X) \triangleq (\bigcup\{\Phi \sqsupset \Psi \mid \Psi \in X\} \in \wp_{\langle\omega\rangle}(\mathrm{Seq}_{\Sigma}^{[(\omega\{\setminus 1\})\vdash](\omega(\setminus 1))})$.

A (purely) multi-conclusion [{purely} multi-premise] sequent Σ-calculus $\mathcal{G}$ is said to be $\langle$*deductively*$\rangle$ *multiplicative*, provided, for every (purely) multi-conclusion [{purely} multi-premise] sequent Σ-rule $X|\Phi$ $\langle$derivable$\rangle$ in $\mathcal{G}$ and each multi-conclusion Σ-sequent Ψ, the rule $(X \uplus \Psi)|(\Phi \uplus \Psi)$ is derivable in $\mathcal{G}$. Using induction on the length of $\mathcal{G}$-derivations, it is routine checking that $\mathcal{G}$ is multiplicative iff it is deductively so.

Theorem 3.11 (cf. the proof of Theorem 4.2 of Pynko (1999))**.** *Let $\mathcal{G}$ be a $\langle$multiplicative$\rangle$ (purely) multi-conclusion [{purely} multi-premise] sequent Σ-calculus with basic structural rules and Cut$\langle$/Reflexivity$\rangle$ and $(X \cup \{\Phi, \Psi\}) \subseteq \mathrm{Seq}_{\Sigma}^{[(\omega\{\setminus 1\})\vdash](\omega(\setminus 1))}$. Then, $\Psi \in \mathrm{Cn}_{\mathcal{G}}(X \cup \{\Phi\}) \Leftarrow \langle / \Rightarrow\rangle (\Phi \sqsupset \Psi) \subseteq \mathrm{Cn}_{\mathcal{G}}(X)$.*

From the model-theoretic point of view, any Σ-sequent $\Gamma \vdash \Delta$ is treated as the first-order basic clause (viz., disjunct) $\bigvee(\neg[\mathrm{img}\,\Gamma] \cup (\mathrm{img}\,\Delta))$ of the signature $\Sigma \cup \{D\}$ under the notorious identification of any Σ-formula φ with the first-order atomic formula $D(\varphi)$, any sequent Σ-rule being interpreted as the universal closure of the implication of its premises (under the natural identification of any finite set X of first-order formulas with $\bigwedge X$ we follow tacitly as well) and its conclusion, in which case sequent Σ-calculi become universal first-order theories. (In this way, sequent disjunction/implication corresponds to the usual disjunction/implication.) This fits the standard matrix interpretation of sequents equally adopted in Pynko (1999) and Pynko (2004).

4. Basic disjunctive calculi

4.1. The Hilbert-style calculus

By $\mathcal{D}_{\veebar}$ we denote the Σ-calculus constituted by the following Σ-rules:

$$\begin{array}{cccc} D_1 & D_2 & D_3 & D_4 \\[1ex] \dfrac{x_0 \veebar x_0}{x_0} & \dfrac{x_0}{x_0 \veebar x_1} & \dfrac{(x_0 \veebar x_1) \veebar x_2}{(x_1 \veebar x_0) \veebar x_2} & \dfrac{(x_0 \veebar (x_1 \veebar x_2)) \veebar x_3}{((x_0 \veebar x_1) \veebar x_2) \veebar x_3} \end{array}$$

Lemma 4.1. *Let C be a Σ-logic, $\mathcal{R} = (\Gamma|\phi)$ a Σ-rule and $v \in (V_\omega \setminus \mathrm{Var}(\mathcal{R}))$. Suppose (2.3) and (2.5) hold and $\mathcal{R} \veebar v$ is satisfied in C. Then, so is $\mathcal{R}$ itself.*

Proof. First, by (2.3), we have $(\Gamma \veebar \phi) \subseteq C(\Gamma)$. Then, applying $(\mathcal{R} \veebar v)[v/\phi]$, by the structurality of C, we get $(\phi \veebar \phi) \in C(\Gamma)$. Finally, (2.5) completes the argument. □

Taking D_1 and D_2 into account and applying Lemma 4.1 with $C = \mathrm{Cn}_{\mathcal{D}_{\veebar}}$ to both D_3 and D_4, we immediately get:

Corollary 4.2. *The following rules are derivable in $\mathcal{D}_{\veebar}$:*

$$\frac{x_0 \veebar x_1}{x_1 \veebar x_0}, \tag{4.1}$$

$$\frac{x_0 \veebar (x_1 \veebar x_2)}{(x_0 \veebar x_1) \veebar x_2}. \tag{4.2}$$

Lemma 4.3. *The following rules are derivable in $\mathcal{D}_{\veebar}$:*

$$\frac{(x_0 \veebar x_1) \veebar x_2}{x_0 \veebar (x_1 \veebar x_2)}, \tag{4.3}$$

$$\frac{(x_0 \veebar x_0) \veebar x_1}{x_0 \veebar x_1}, \tag{4.4}$$

$$\frac{x_0 \veebar x_2}{(x_0 \veebar x_1) \veebar x_2}. \tag{4.5}$$

Proof. First, in view of Corollary 4.2, (4.3) is by the following $\mathrm{Cn}_{\mathcal{D}_{\veebar}}$-derivation:

(1) $(x_0 \veebar x_1) \veebar x_2$ — hypothesis;
(2) $(x_1 \veebar x_0) \veebar x_2$ — D_3: 1;
(3) $x_2 \veebar (x_1 \veebar x_0)$ — (4.1)$[x_0/(x_1 \veebar x_0), x_1/x_2]$: 2;
(4) $(x_2 \veebar x_1) \veebar x_0$ — (4.2)$[x_0/x_2, x_2/x_0]$: 3;
(5) $(x_1 \veebar x_2) \veebar x_0$ — $D_3[x_0/x_2, x_2/x_0]$: 4;
(6) $x_0 \veebar (x_1 \veebar x_2)$ — (4.1)$[x_0/(x_1 \veebar x_0), x_1/x_0]$: 5.

Then, in view of Corollary 4.2, (4.4) is by the following $\mathrm{Cn}_{\mathcal{D}_{\veebar}}$-derivation:

(1) $(x_0 \veebar x_0) \veebar x_1$ — hypothesis;
(2) $x_0 \veebar (x_0 \veebar x_1)$ — (4.3)$[x_1/x_0, x_2/x_1]$: 1;
(3) $(x_0 \veebar x_1) \veebar x_0$ — (4.1)$[x_1/(x_0 \veebar x_1)]$: 2;
(4) $((x_0 \veebar x_1) \veebar x_0) \veebar x_1$ — $D_2[x_0/((x_0 \veebar x_1) \veebar x_0)]$: 3;
(5) $(x_0 \veebar x_1) \veebar (x_0 \veebar x_1)$ — (4.3)$[x_0/(x_0 \veebar x_1), x_1/x_0, x_1/x_2]$: 4;
(6) $(x_0 \veebar x_1)$ — $D_1[x_0/(x_0 \veebar x_1)]$: 5.

Finally, in view of Corollary 4.2, (4.5) is by the following $\mathrm{Cn}_{\mathcal{D}_{\veebar}}$-derivation:

(1) $x_0 \veebar x_2$ — hypothesis;
(2) $(x_0 \veebar x_2) \veebar x_1$ — $D_2[x_0/(x_0 \veebar x_2)]$: 1;
(3) $x_0 \veebar (x_2 \veebar x_1)$ — (4.3)$[x_1/x_2, x_2/x_1]$: 2;
(4) $(x_2 \veebar x_1) \veebar x_0$ — (4.1)$[x_1/(x_2 \veebar x_1)]$: 3;
(5) $x_2 \veebar (x_1 \veebar x_0)$ — (4.3)$[x_0/x_2, x_2/x_0]$: 4;
(6) $(x_1 \veebar x_0) \veebar x_2$ — (4.1)$[x_0/x_2, x_1/(x_1 \veebar x_0)]$: 5;
(7) $(x_0 \veebar x_1) \veebar x_2$ — $D_3[x_0/x_1, x_1/x_0]$: 6. □

Theorem 4.4. $\mathrm{Cn}_{\mathcal{D}_{\veebar}}$ *is* $\veebar$*-disjunctive.*

Proof. With using Theorem 3.3. First, by D_1, D_2, Corollary 4.2 and Lemma 4.3(4.3), (2.3), (2.5), (2.6) and (2.7) hold for $C \triangleq \mathrm{Cn}_{\mathcal{D}_{\veebar}}$.

Next, consider any $\sigma \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{Fm}^{\omega}_{\Sigma})$, any $\psi \in \mathrm{Fm}^{\omega}_{\Sigma}$ and any $i \in (5 \setminus 1)$. The case, when $i \notin 3$, is due to Lemma 3.4 with $v = x_{i-1}$ and such $\mathcal{R}$ that $D_i = (\mathcal{R} \veebar v)$. Otherwise, we have $\mathrm{Var}(D_i) = V_i \not\ni x_i$. Then, by Lemma 4.3(4.4)/(4.5), $D_i \veebar x_i$ is derivable in $\mathcal{D}_{\veebar}$. Let $\varsigma \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{Fm}^{\omega}_{\Sigma})$ extend $(\sigma \restriction V_{\omega \setminus \{i\}}) \cup [x_i/\psi]$, in which case $\varsigma(D_i) = \sigma(D_i)$, and so, by the structurality of C, we eventually conclude that $(\sigma(D_i) \veebar \psi) = (\varsigma(D_i) \veebar \varsigma(x_i)) = \varsigma(D_i \veebar x_i)$ is derivable in $\mathcal{D}_{\veebar}$, as required. □

The following auxiliary observation has proved quite useful for reducing the number of rules of calculi to be constructed in Section 6 according to the universal method to be elaborated in Section 5:

Corollary 4.5. *Let* $\phi, \psi, \varphi \in \mathrm{Fm}^{\omega}_{\Sigma}$, $v \in (V_{\omega} \setminus (\bigcup \mathrm{Var}[\{\phi, \psi, \varphi\}]))$ *and* $\mathcal{C} \supseteq \mathcal{D}_{\veebar}$ *a* Σ*-calculus. Then, the rules* $(\phi \veebar v) | (\varphi \vee v)$ *and* $(\psi \veebar v) | (\varphi \vee v)$ *are both derivable in* $\mathcal{C}$ *iff the rule* $((\phi \veebar \psi) \vee v) | (\varphi \vee v)$ *is so.*

Proof. First of all, by Theorem 4.4, $C \triangleq \mathrm{Cn}_{\mathcal{D}_{\veebar}} \subseteq C' \triangleq \mathrm{Cn}_{\mathcal{C}}$ is $\veebar$-disjunctive, and so, by Lemma 2.1, is δ-multiplicative. Then, the "if" part is by (2.3), (2.4) and (2.1) with $X = \varnothing$, $a = v$ and $Y = \{\phi/\psi\}$, for $C \subseteq C'$. Conversely, assume both $(\varphi \veebar v) \in C'(\phi \veebar v)$ and $(\varphi \veebar v) \in C'(\psi \veebar v)$, applying $[v/(\psi \veebar v)]$ and $[v/(v \veebar \varphi)]$, respectively, to which, by the structurality of C', we get both $(\varphi \veebar (\psi \veebar v)) \in C'(\phi \veebar (\psi \veebar v))$ and $(\varphi \veebar (v \veebar \varphi)) \in C'(\psi \veebar (v \veebar \varphi))$. In this way, as $C \subseteq C'$, by (2.1) with $X = \varnothing$, $a = v$ and $Y = \{\varphi \veebar \varphi\}$, (2.5), (2.6) and (2.7), we eventually get $(\varphi \veebar v) \in C'((\varphi \veebar \varphi) \veebar v) = C'(v \veebar (\varphi \veebar \varphi)) = C'((v \veebar \varphi) \veebar \varphi) = C'(\varphi \veebar (v \veebar \varphi)) \subseteq C'(\psi \veebar (v \veebar \varphi)) = C'((\psi \veebar v) \veebar \varphi) = C'(\varphi \veebar (\psi \veebar v)) \subseteq C'(\phi \veebar (\psi \veebar v)) = C'((\phi \veebar \psi) \veebar v)$, as required. □

4.2. Single- versus multi-conclusion sequent calculi

Let $\mathcal{G}^{\alpha}_{\veebar}$, where $\alpha \subseteq \omega$, be the α-conclusion sequent Σ-calculus constituted by structural α-conclusion sequent $\varnothing$-rules and the following α-conclusion sequent Σ-rules:

$$\begin{array}{cc} G_l & G_r \\ \dfrac{\Gamma, x_0 \vdash \Delta \quad \Gamma, x_1 \vdash \Delta}{\Gamma, (x_0 \veebar x_1) \vdash \Delta} & \dfrac{\Gamma \vdash \Omega, x_k}{\Gamma \vdash \Omega, (x_0 \veebar x_1)} \end{array}$$

where $k \in 2$ and $\Gamma, \Delta, \Omega \in V_\omega^*$ such that $(\mathrm{dom}\,\Delta), ((\mathrm{dom}\,\Omega) + 1) \in \alpha$.

Lemma 4.6. *Let $\psi \in \mathrm{Fm}^{\omega}_{\veebar}$ and $v \in \mathrm{Var}(\psi)$. Suppose $1 \in \alpha$. Then, $v \vdash \psi$ is derivable in $\mathcal{G}^{\alpha}_{\veebar}$.*

Proof. By induction on construction of ψ. For consider the following complementary cases:

(1) $\psi \in V_\omega$.
Then, $\mathrm{Var}(\psi) = \{\psi\} \ni v$, in which case $\psi = v$, and so the Reflexivity axiom completes the argument.

(2) $\psi \notin V_\omega$.
Then, $\psi = (\varphi_0 \veebar \varphi_1)$, for some $\varphi_0, \varphi_1 \in \mathrm{Fm}^{\omega}_{\veebar}$, in which case $v \in \mathrm{Var}(\psi) = (\bigcup_{k\in 2} \mathrm{Var}(\varphi_k))$, and so $v \in \mathrm{Var}(\varphi_k)$, for some $k \in 2$. Hence, by induction hypothesis, $v \vdash \varphi_k$ is derivable in $\mathcal{G}^{\alpha}_{\veebar}$. In this way, G_r completes the argument. □

Corollary 4.7. *Let $\phi, \psi \in \mathrm{Fm}^{\omega}_{\veebar}$. Suppose $\mathrm{Var}(\phi) \subseteq \mathrm{Var}(\psi)$ and $1 \in \alpha$. Then, $\phi \vdash \psi$ is derivable in $\mathcal{G}^{\alpha}_{\veebar}$.*

Proof. By induction on construction of ϕ. For consider the following complementary cases:

(1) $\phi \in V_\omega$.
Then, $\mathrm{Var}(\psi) \supseteq \mathrm{Var}(\phi) = \{\phi\}$, in which case $\phi \in \mathrm{Var}(\psi)$, and so Lemma 4.6 completes the argument.

(2) $\phi \notin V_\omega$.
Then, $\phi = (\varphi_0 \veebar \varphi_1)$, for some $\varphi_0, \varphi_1 \in \mathrm{Fm}^{\omega}_{\veebar}$, in which case $\mathrm{Var}(\psi) \supseteq \mathrm{Var}(\phi) = (\bigcup_{k\in 2} \mathrm{Var}(\varphi_k))$, and so $\mathrm{Var}(\psi) \supseteq \mathrm{Var}(\varphi_k)$, for each $k \in 2$. Hence, by induction hypothesis, $\varphi_k \vdash \psi$ is derivable in $\mathcal{G}^{\alpha}_{\veebar}$, for every $k \in 2$. Thus, G_l completes the argument. □

Let $\tau_{\veebar} : \mathrm{Seq}_{\Sigma}^{\omega} \to \mathrm{Seq}_{\Sigma}^{2}$ be defined as follows:

$$\tau_{\veebar}(\Gamma \vdash \Delta) \triangleq \begin{cases} \Gamma \vdash \Delta & \text{if } \Delta = \varnothing, \\ \Gamma \vdash (\veebar\Delta) & \text{otherwise,} \end{cases}$$

for all $(\Gamma \vdash \Delta) \in \mathrm{Seq}_{\Sigma}^{\omega}$, in which case:

$$\sigma(\tau_{\veebar}(\Gamma \vdash \Delta)) = \tau_{\veebar}(\sigma(\Gamma \vdash \Delta)). \tag{4.6}$$

Lemma 4.8. *For every $\mathcal{R} \in \mathcal{G}_{\veebar}^{\omega[\setminus 1]}$, $\tau_{\veebar}(\mathcal{R})$ is derivable in $\mathcal{G}_{\veebar}^{2[\setminus 1]}$.*

Proof. Consider the following exhaustive cases:

(1) $\mathcal{R}$ is either G_l or the Reflexivity axiom or a left-side basic structural rule or a Cut with $\Delta = \varnothing$.
Then, $\tau_{\veebar}(\mathcal{R})$ is a substitutional Σ-instance of a rule in $\mathcal{G}_{\veebar}^{2[\setminus 1]}$, and so is derivable in it.

(2) $\mathcal{R}$ is either G_r or a right-side basic structural rule.
Then, $\tau_{\veebar}(\mathcal{R})$ is of the form

$$\frac{\Lambda \vdash \phi}{\Lambda \vdash \psi},$$

where $\Lambda \in V_{\omega}^{*}$ and $\phi, \psi \in \mathrm{Fm}_{\veebar}^{\omega}$, while $\mathrm{Var}(\phi) \subseteq \mathrm{Var}(\psi)$, in which case Corollary 4.7 and Cut complete the argument.

(3) $\mathcal{R}$ is a Cut with $\Delta \neq \varnothing$.
Then, $\tau_{\veebar}(\mathcal{R})$ is as follows:

$$\frac{\Lambda, \Gamma \vdash (\phi \veebar x_0) \quad \Gamma, x_0 \vdash \psi}{\Lambda, \Gamma \vdash \psi},$$

where $\phi \triangleq (\veebar\Delta) \in \mathrm{Fm}_{\veebar}^{\omega}$ and $\psi \triangleq (\veebar(\Delta, \Theta)) \in \mathrm{Fm}_{\veebar}^{\omega}$, in which case $\mathrm{Var}(\phi) \subseteq \mathrm{Var}(\psi)$, and so, by Corollary 4.7, $\phi \vdash \psi$ is derivable in $\mathcal{G}_{\veebar}^{2[\setminus 1]}$, and so is $\Gamma, \phi \vdash \psi$, by basic structural rules. Hence, by G_l, the rule $(\Gamma, x_0 \vdash \psi)|(\Gamma, (\phi \veebar x_0) \vdash \psi)$ is derivable in $\mathcal{G}_{\veebar}^{2[\setminus 1]}$. Thus, Cut completes the argument. □

With using induction on the length of $(\mathcal{G}_{\veebar}^{\omega[\setminus 1]} \cup \mathcal{A})$-derivations, by (4.6), Lemma 4.8 and the structurality of the consequence of any calculus, we immediately get:

Theorem 4.9. *Let $(\mathcal{A} \cup \{\Phi\}) \subseteq \mathrm{Seq}_{\Sigma}^{\omega[\setminus 1]}$. Suppose Φ is derivable in $\mathcal{G}_{\veebar}^{\omega[\setminus 1]} \cup \mathcal{A}$. Then, $\tau_{\veebar}(\Phi)$ is derivable in $\mathcal{G}_{\veebar}^{2[\setminus 1]} \cup \tau_{\veebar}[\mathcal{A}]$.*

Though the converse holds as well, because any [purely] milti-conclusion sequent Ψ and $\tau_{\veebar}[\Psi]$ are derivable from one another in $\mathcal{G}_{\veebar}^{\omega[\setminus 1]} \supseteq \mathcal{G}_{\veebar}^{2[\setminus 1]}$, this point is no matter for our further argumentation.

5. Main general results

Fix any finite $\veebar$-disjunctive Σ-matrix $\mathcal{A}$ with a finite equality determinant $\Upsilon \subseteq \mathrm{Var}^{-1}[\{V_1\}]$ containing x_0 to be supposed to be totally-ordered by some $\lesssim$, x_0 being its greatest element. Let Σ_0 be the set of nullary elements of Σ. Given any $X \subseteq (V_\omega \cup \Sigma_0)$, put $\Upsilon[X] \triangleq \{\upsilon(x) \mid \upsilon \in \Upsilon, x \in X\}$.

To simplify further notations, we adopt the following "sign" one: given any $\Gamma \in (\mathrm{Fm}^\omega_\Sigma)^*$ and any $\Bbbk \in 2$, put $(\Bbbk : \Gamma) \triangleq \{\langle \Bbbk, \Gamma\rangle, \langle 1-\Bbbk, \varnothing\rangle\} \in \mathrm{Seq}^\omega_\Sigma$.

Following Pynko (2004), elements of $\Upsilon \times \Sigma$ are referred to as $\langle\Upsilon,\Sigma\rangle$-*types*, a $\langle\Upsilon,\Sigma\rangle$-type $\langle\upsilon, F\rangle$, where F is of arity $n \in \omega$, being said to be Υ-*complex*, whenever both $n \neq 0$ and $(n = 1) \Rightarrow (\upsilon(F(x_0)) \notin \Upsilon)$. Then, extending Pynko (2004), a Σ-*sequential* Υ-*table for* $\mathcal{A}$ is any couple $\mathcal{T}$ of functions with domain $\Upsilon \times \Sigma$, in which case we set $(\lambda/\rho)_\mathcal{T} \triangleq \pi_{0/1}(\mathcal{T})$ to adapt conventions adopted in Pynko (2004), such that, for all $\Bbbk \in 2$ and each $\langle\upsilon, F\rangle \in (\Upsilon \times \Sigma)$, where F is of arity $n \in \omega$, $\pi_\Bbbk(\mathcal{T})(\upsilon, F) \in \wp_\omega(\Upsilon[V_n]^*)^2$ has solely injective elements and is equivalent to $\Bbbk : \upsilon(F(x_i)_{i\in n})$ with respect to $\mathcal{A}$:

$$\mathcal{A} \models \langle\forall x_i\rangle_{i\in n}((\Bbbk : \upsilon(F(x_i)_{i\in n})) \leftrightarrow \pi_\Bbbk(\mathcal{T})(\upsilon, F)), \tag{5.1}$$

in which case every element of $(\boldsymbol{\lambda}/\boldsymbol{\rho})_\mathcal{T}(\upsilon, F) \triangleq ((\rho/\lambda)_\mathcal{T}(\upsilon, F) \uplus \{(0/1) : \upsilon(F(x_i)_{i\in n})\})$ is true in $\mathcal{A}$, that exists, by the constructive proof of Theorem 1 of Pynko (2004), though not being unique, generally speaking.

Example 5.1. When $\upsilon = x_0$ and $F = \veebar$, in which case $\veebar$ is a primary binary connective of Σ, and so $\langle\upsilon, F\rangle$ is Υ-complex, one can always take $\lambda_\mathcal{T}(\upsilon, F) = \{x_0 \vdash; x_1 \vdash\}$ and $\rho_\mathcal{T}(\upsilon, F) = \{\vdash x_0, x_1\}$ to satisfy (5.1), in which case $\boldsymbol{\lambda}_\mathcal{T}(\upsilon, F) = \{(x_0 \veebar x_1) \vdash x_0, x_1\}$ and $\boldsymbol{\rho}_\mathcal{T}(\upsilon, F) = \{x_0 \vdash (x_0 \veebar x_1); x_1 \vdash (x_0 \veebar x_1)\}$, and so their elements are all derivable in $\mathcal{G}^\omega_\veebar$. □

Then, let $\mathcal{A}'$ be the set of all elements of $\boldsymbol{\lambda}_\mathcal{T}(\upsilon, F) \cup \boldsymbol{\rho}_\mathcal{T}(\upsilon, F)$, for all Υ-complex $\langle\Upsilon,\Sigma\rangle$-types $\langle\upsilon, F\rangle$ but $\langle x_0, \veebar\rangle$, in case $\veebar \in \Sigma$ is primary.

Next, let $\mathcal{A}''$ be the set containing, for each $c \in \Sigma_0$ and every $\upsilon \in \Upsilon$, exactly that of the either axioms $(0/1) : \upsilon(c)$, which is true in $\mathcal{A}$.

Given any $\overline{\Omega} \in \mathrm{Seq}^\omega_\Sigma(V_1)^*$, set

$$(\amalg\overline{\Omega}) \triangleq \begin{cases} \vdash & \text{if } \overline{\Omega} = \varnothing, \\ \uplus\langle\Omega_i[x_0/x_i]\rangle_{i\in n} & \text{otherwise.} \end{cases}$$

Further, the finite set $\mathrm{Ax}(\Upsilon)$ of all disjoint injective elements of $((\Upsilon)^*)^2$ with monotonic sides is easily seen to be *partially*-ordered by $\sqsubseteq_{[1]}$. Given any set S of Σ-matrices, let $\mathrm{Ax}(\mathsf{S})$ be the set of all elements of $\mathrm{Ax}(\Upsilon)$ true in S, $\mathrm{Ax}^*_{[1]}(\mathsf{S}) \triangleq \min_{\sqsubseteq_{[1]}}(\mathrm{Ax}(\mathsf{S}))$ and $\rho_\mathsf{S} : \mathrm{Seq}^\omega_\Sigma \to \wp(\mathsf{S}), \Phi \mapsto (\mathsf{S} \cap \mathrm{Mod}(\Phi))$.

Proposition 5.2. *Let $n \in \omega$ and $\overline{\Omega} \in \mathrm{Seq}^\omega_\Sigma(V_1)^n$. Then,* $\mathrm{Mod}(\overline{\Omega}) = (\bigcup_{i\in n}\mathrm{Mod}(\Omega_i))$. *In particular, $\amalg\overline{\Omega}$ is true in a set S of Σ-matrices iff $\mathrm{img}(\rho_\mathsf{S} \circ \overline{\Omega})$ is a covering of S.*

Proof. The case, when $n = 0$, is by the fact that $\mathrm{Mod}(\vdash) = \varnothing$. Otherwise, the inclusion from right to left is immediate. Conversely, consider any Σ-matrix $\mathcal{D} \notin (\bigcup_{i\in n}\mathrm{Mod}(\Omega_i))$, in which case, for every $i \in n$, there is some $d_i \in D$ such that $\mathcal{D} \not\models \Omega_i[x_0/d_i]$, and so $\mathcal{D} \not\models (\amalg\overline{\Omega})[x_i/d_i]_{i\in n}$, as required. □

Furthermore, members of $\mathbf{S}(\mathcal{A})$ are uniquely determined by and so naturally identified with the carriers of their underlying algebras. Let $\mathsf{A} \subseteq \mathbf{S}(\mathcal{A})$, in which case $m \triangleq |\mathsf{A}| \in \omega$, $\overline{\mathcal{B}} : m \to \mathsf{A}$ any enumeration of A and $\mathcal{M}$ any MCES for A. Then, every element of the finite set $\mathcal{A}'''_{[1]}(\mathcal{M}) \triangleq \{\amalg\overline{\Omega} \mid \overline{\mathsf{X}} \in \mathcal{M}, n = (\mathrm{dom}\,\overline{X}), \overline{\Omega} \in \prod_{i\in n} \min_{\sqsubseteq_{[1]}}(\mathrm{Ax}(\Upsilon) \cap \rho_{\mathsf{A}}^{-1}[\{\mathsf{X}_i\}])\}$ is true in A, by Proposition 5.2.

Finally, every element of $\mathcal{A}_{[1]}(\mathcal{M}) \triangleq (\mathcal{A}' \cup \mathcal{A}'' \cup \mathcal{A}'''_{[1]}(\mathcal{M}))$ is true in A. Moreover, $\mathcal{A}_{[1]}(\mathcal{M})$ is finite, whenever Σ is so.

Lemma 5.3. *Any multi-conclusion Σ-sequent Φ is true in A iff it is derivable in $\mathcal{G}^{\omega}_{\veebar} \cup \mathcal{A}_{[1]}(\mathcal{M})$.*

Proof. The "if" part is by the fact that every element of $\mathcal{A}_{[1]}(\mathcal{M})$ is true in A, while any $\veebar$-disjunctive Σ-matrix (in particular, a submatrix of $\mathcal{A}$) is a model of $\mathcal{G}^{\omega}_{\veebar}$. The converse is proved by induction on $\partial(\Phi) \in \omega$, following the proof of Theorem 2 of Pynko (2004). For suppose Φ is true in $\mathsf{A}_{[1]}$. First, assume $\partial(\Phi) = 0$, that is, $\Phi \in (\Upsilon[V_\omega \cup \Sigma_0]^*)^2$. Then, the case, when either Φ is not disjoint or $\Psi \sqsubseteq_1 \Phi$, for some $\Psi \in \mathcal{A}''$, is by Reflexivity, basic structural rules and the structurality of the consequence of any calculus. The opposite case is by basic structural rules and the structurality of the consequence of any calculus as well the following two claims:

Claim 5.4. *Let $\Phi \in (\Upsilon[V_\omega \cup \Sigma_0]^*)^2$. Suppose it is disjoint and true in A, while $\Psi \sqsubseteq_1 \Phi$, for no $\Psi \in \mathcal{A}''$. Then, there are some $\overline{\Omega} \in \prod_{j\in m} \mathrm{Ax}(\mathcal{B}_j)$ and some $\bar{v} \in V_\omega^m$ such that, for every $j \in m$, $(\Omega_j[x_0/v_j]) \sqsubseteq_1 \Phi$, in which case $((\amalg\overline{\Omega})[x_j/v_j]_{j\in m}) \sqsubseteq_1 \Phi$, and so $(\amalg\overline{\Omega}) \sqsubseteq \Phi$.*

Proof. By the item 4 of the proof of Theorem 2 of Pynko (2004), because, in that case, for each $j \in m$, Φ is true in $\mathcal{B}_j \in \mathsf{A}$ inheriting the equality determinant Υ. □

Claim 5.5. *Let $\overline{\Omega} \in \mathrm{Ax}(\Upsilon)^*$. Suppose $\Omega \triangleq (\amalg\overline{\Omega})$ is true in A (that is, $C \triangleq \mathrm{img}(\rho_{\mathsf{A}} \circ \overline{\Omega})$ is a covering of A; cf. Proposition 5.2), Then, there is some $\Xi \in \mathcal{A}'''_{[1]}(\mathcal{M})$ such that $\Xi \sqsubseteq \Omega$.*

Proof. In that case, as A is finite, C includes a minimal covering M of A. Let $\overline{\mathsf{X}}$ be the unique enumeration of M belonging to $\mathcal{M}$. Consider any $i \in n \triangleq |M|$, in which case $\mathsf{X}_i \in M \subseteq C$, and so there is some $j_i \in (\mathrm{dom}\,\overline{\Omega})$ such that $\mathsf{X}_i = \rho_{\mathsf{A}}(\Omega_{j_i})$. Then, $\Omega_{j_i} \in \mathrm{Ax}(\mathsf{X}_i)$, in which case there is some $\Xi_i \in \mathrm{Ax}^*_{[1]}(\mathsf{X}_i)$ such that $\Xi_i \sqsubseteq_{[1]} \Omega_{j_i}$, and so there is some $\varphi_i[= x_0] \in \mathrm{Fm}^1_\Sigma$ such that $(\Xi_i[x_0/\varphi_i]) \sqsubseteq_1 \Omega_{j_i}$. And what is more, $\rho_{\mathsf{A}}(\Xi_i) \subseteq \rho_{\mathsf{A}}(\Omega_{j_i}) = \mathsf{X}_i \subseteq \rho_{\mathsf{A}}(\Xi_i)$, in which case $\rho_{\mathsf{A}}(\Xi_i) = \mathsf{X}_i$, and so $\Xi_i \in (\mathrm{Ax}(\Upsilon) \cap \rho_{\mathsf{A}}^{-1}[\{\mathsf{X}_i\}]) \subseteq \mathrm{Ax}(\mathsf{X}_i)$, in which case $\Xi_i \in \min_{\sqsubseteq_{[1]}}(\mathrm{Ax}(\Upsilon) \cap \rho_{\mathsf{A}}^{-1}[\{\mathsf{X}_i\}])$, for $\Xi_i \in \mathrm{Ax}^*_{[1]}(\mathsf{X}_i)$, and so $\Xi \triangleq (\amalg\overline{\Xi}) \in \mathcal{A}'''_{[1]}(\mathcal{M})$. After all, for every $i \in n$, we have $((\Xi_i[x_0/\varphi_i])[x_0/x_{j_i}]) \sqsubseteq_1 (\Omega_{j_i}[x_0/x_{j_i}]) \sqsubseteq_1 \Omega$, in which case we get $(\Xi[x_i/\varphi_i(x_{j_i})]_{i\in n}) \sqsubseteq_1 \Omega$, and so $\Xi \sqsubseteq \Omega$, as required. □

Next, consider any complex $\langle\Upsilon, \Sigma\rangle$-type $\langle v, F\rangle$. We start from proving the fact that the rule:

$$\frac{\lambda_{\mathcal{T}}(v, F) \cup \rho_{\mathcal{T}}(v, F)}{\vdash} \tag{5.2}$$

is derivable in $\mathcal{G}^{\omega}_{\veebar} \cup \mathcal{A}_{[1]}(\mathcal{M})$. Let $n \triangleq |\lambda_{\mathcal{T}}(v, F) \cup \rho_{\mathcal{T}}(v, F)| \in \omega$. Take any bijection $\overline{\Psi} : n \to (\lambda_{\mathcal{T}}(v, F) \cup \rho_{\mathcal{T}}(v, F))$. Then, by (5.1), the rule (5.2) is true in A, and so are all axioms in $(\overline{\Psi} \sqsupset \{\vdash\}) \subseteq (\Upsilon[V_\omega]^*)^2 \subseteq \partial^{-1}[\{0\}]$. Therefore, taking the above argumentation of the case, when $\partial(\Phi) = 0$, into account, all axioms in $(\overline{\Psi} \sqsupset \{\vdash\})$ are derivable in $\mathcal{G}^{\omega}_{\underline{\vee}} \cup \mathcal{A}_{[1]}(\mathcal{M})$. Hence, applying n times Theorem 3.11, we conclude that the rule (5.2) is derivable in $\mathcal{G}^{\omega}_{\underline{\vee}} \cup \mathcal{A}_{[1]}(\mathcal{M})$. Moreover, $\mathcal{G}^{\omega}_{\underline{\vee}} \cup \mathcal{A}_{[1]}(\mathcal{M})$ is clearly multiplicative, and so deductively so. In this way, since every element of $\boldsymbol{(\lambda/\rho)}_{\mathcal{T}}(v, F)$, being in $\mathcal{A}_{[1]}(\mathcal{M})$, unless $v = x_0$ and $F = \underline{\vee}$, is derivable in $\mathcal{G}^{\omega}_{\underline{\vee}} \cup \mathcal{A}_{[1]}(\mathcal{M})$, in view of Example 5.1, taking basic structural rules into account, we see that the rule

$$\frac{(\lambda/\rho)_{\mathcal{T}}(v, F)}{(0/1) : v(F(x_i)_{i\in n})}$$

is derivable in $\mathcal{G}^{\omega}_{\underline{\vee}} \cup \mathcal{A}_{[1]}(\mathcal{M})$. Thus, in view of the deductive multiplicativity of $\mathcal{G}^{\omega}_{\underline{\vee}} \cup \mathcal{A}_{[1]}(\mathcal{M})$ as well as the structurality of the consequence of any calculus, taking basic structural rules into account, we see that all those rules, which belong to Definition 1(v) of Pynko (2004), are derivable in $\mathcal{G}^{\omega}_{\underline{\vee}} \cup \mathcal{A}_{[1]}(\mathcal{M})$. In this way, the case, when $\partial(\Phi) \neq 0$, is due to the last paragraph of the proof of Theorem 2 of Pynko (2004), as required. □

Corollary 5.6. *Let $\mathcal{S} \subseteq \mathrm{Seq}^{\omega}_{\Sigma}$. Then, any multi-conclusion Σ-sequent Φ is true in $\mathbf{S}(\mathsf{A}) \cap \mathrm{Mod}(\mathcal{S})$ iff it is derivable in $\mathcal{G}^{\omega}_{\underline{\vee}} \cup \mathcal{A}_{[1]}(\mathcal{M}) \cup \mathcal{S}$.*

Proof. The "if" part is by that of Lemma 5.3. Conversely, assume Φ is true in $\mathbf{S}(\mathsf{A}) \cap \mathrm{Mod}(\mathcal{S})$. Consider any $\mathcal{B} \in \mathsf{A}$ and any $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{B})$ such that $\mathcal{B} \models \Psi[h]$, for all $\Psi \in \mathrm{SI}_{\Sigma}(\mathcal{S})$. Then, $D \triangleq (\mathrm{img}\, h)$ forms a subalgebra of $\mathfrak{B}$, while $\mathcal{D} \triangleq (\mathcal{B} \upharpoonright D) \in \mathbf{S}(\mathsf{A})$, whereas $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{D})$ is surjective. Consider any $g \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{D})$ and any $\Omega \in \mathcal{S}$. Then, there is some Σ-substitution σ such that $g = (h \circ \sigma)$, in which case $\sigma(\Omega) \in \mathrm{SI}_{\Sigma}(\mathcal{S})$, and so $\mathcal{D} \models \Omega[g]$, for $\mathcal{B} \models \sigma(\Omega)[h]$. Thus, $\mathcal{D} \in \mathrm{Mod}(\mathcal{S})$, in which case Φ is true in $\mathcal{D}$, and so, in particular, $\mathcal{B} \models \Phi[h]$. In this way, first-order formulas of $\mathrm{SI}_{\Sigma}(\mathcal{S}) \cup \{\neg\Phi\}$ are *collectively* true in no member of A under any *common* assignment. On the other hand, as both A and all members of it are finite, any ultra-product of any tuple constituted by its members is isomorphic to one of them. Therefore, by the Compactness Theorem Mal'cev (1965), there is some $\overline{\Psi} \in \mathrm{SI}_{\Sigma}(\mathcal{S})^*$ such that $\overline{\Psi} \sqsupset \{\Phi\}$ is true in A. Hence, by Lemma 5.3, all elements of $\overline{\Psi} \sqsupset \{\Phi\}$ are derivable in $\mathcal{G}^{\omega}_{\underline{\vee}} \cup \mathcal{A}_{[1]}(\mathcal{M})$, in which case, applying $(\mathrm{dom}\, \overline{\Psi})$ times Theorem 3.11, we eventually conclude that $(\mathrm{img}\, \overline{\Psi})|\Phi$ is derivable in $\mathcal{G}^{\omega}_{\underline{\vee}} \cup \mathcal{A}_{[1]}(\mathcal{M})$, and so Φ is derivable in $\mathcal{G}^{\omega}_{\underline{\vee}} \cup \mathcal{A}_{[1]}(\mathcal{M}) \cup \mathcal{S}$. □

Given any $\mathcal{S} \subseteq \mathrm{Seq}^{\omega}_{\Sigma}$, set $\mathcal{S}_{\setminus 1} \triangleq ((\mathcal{B} \cap \mathrm{Seq}^{\omega\setminus 1}_{\Sigma}) \cup \{(\sigma_{+1} \circ \Gamma) \vdash x_0 \mid \Gamma \in (\mathrm{Fm}^{\omega}_{\Sigma})^*, (\Gamma\vdash) \in \mathcal{B}\}) \subseteq \mathrm{Seq}^{\omega\setminus 1}_{\Sigma}$.

Lemma 5.7. *Let $\mathcal{S} \subseteq \mathrm{Seq}^{\omega}_{\Sigma}$ and $\mathcal{C}$ a [consistent] Σ-matrix. Then, $\mathcal{C} \in \mathrm{Mod}(\mathcal{S}_{\setminus 1})$ if[f] $\mathcal{C} \in \mathrm{Mod}(\mathcal{S})$.*

Proof. The "if" part is immediate. [Conversely, consider any $\Gamma \in (\mathrm{Fm}^{\omega}_{\Sigma})^*$ such that $(\sigma_{+1} \circ \Gamma) \vdash x_0$ is true in $\mathcal{C}$ and any $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{C})$. Take any $a \in (C \setminus D^{\mathcal{C}}) \neq \varnothing$. Let $g \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{C})$ extend $[x_{i+1}/h(x_i); x_0/a]_{i\in\omega}$, in which case $h = (g \circ \sigma_{+1})$, and so $\mathcal{C} \models ((\sigma_{+1} \circ \Gamma) \vdash x_0)[g]$ means $\mathcal{C} \models (\Gamma \vdash)[h]$, for $a \notin D^{\mathcal{C}}$. Thus, $\Gamma \vdash$ is true in $\mathcal{C}$.] □

In particular, elements of $\mathcal{A}_{[1]}(\mathcal{M})_{\setminus 1}$ are true in A, for those of $\mathcal{A}_{[1]}(\mathcal{M})$ are so.

Lemma 5.8. *Let $\mathcal{S} \subseteq \mathrm{Seq}^{\omega}_{\Sigma}$. Then, any purely multi-conclusion Σ-sequent is derivable in $\mathcal{G}^{\omega}_{\underline{\vee}} \cup \mathcal{S}$ iff it is derivable in $\mathcal{G}^{\omega\setminus 1}_{\underline{\vee}} \cup \mathcal{S}_{\setminus 1}$.*

Proof. The "if" part is by the inclusion $\mathcal{G}^{\omega\backslash 1}_{\veebar}\cup\mathcal{S}_{\backslash 1}\subseteq\mathrm{Cn}_{\mathcal{G}^{\omega}_{\veebar}\cup\mathcal{S}}$ held by the structurality of the latter and Right Enlargement.

Conversely, consider any $\Phi=(\Gamma\vdash\Delta)\in\mathrm{Seq}^{\omega\backslash 1}_{\Sigma}$ and any $\mathcal{G}^{\omega}_{\veebar}\cup\mathcal{S}$-derivation $\overline{\Psi}$ of it of length $n\in\omega$. Take any $\varphi\in(\mathrm{img}\,\Delta)\neq\varnothing$. Then, in view of right-side basic structural rules, $\langle\langle\Psi_i\uplus(\vdash\varphi)\rangle_{i\in n},\Phi\rangle$ is a $\mathrm{Cn}_{\mathcal{G}^{\omega\backslash 1}_{\veebar}\cup\mathcal{S}_{\backslash 1}}$-derivation of Φ, as required. □

Corollary 5.9. *Let $\mathcal{S}\subseteq\mathrm{Seq}^{\omega}_{\Sigma}$. Then, any [purely] single-conclusion Σ-sequent is true in $\mathsf{A}\cap\mathrm{Mod}(\mathcal{S})$ iff it is derivable in $\mathcal{G}^{2[\backslash 1]}_{\veebar}\cup\tau_{\veebar}[(\mathcal{A}_{(1)}(\mathcal{M})\cup\mathcal{S})_{[\backslash 1]}]$.*

Proof. The "if" part is by the fact that every member of $\mathsf{A}\cap\mathrm{Mod}(\mathcal{S})$, being clearly a $\veebar$-disjunctive model of $(\mathcal{A}_{(1)}(\mathcal{M})\cup\mathcal{S})_{[\backslash 1]}$ [cf. Lemma 5.7], is then a model of $\mathcal{G}^{2[\backslash 1]}_{\veebar}\cup\tau_{\veebar}[(\mathcal{A}_{(1)}(\mathcal{M})\cup\mathcal{S})_{[\backslash 1]}]$. The converse is by Theorem 4.9 and Corollary 5.6 as well as [both Lemma 5.8 and] the diagonality of $\tau_{\veebar}\upharpoonright\mathrm{Seq}^{2[\backslash 1]}_{\Sigma}$. □

Given an axiomatic [finite] purely single-conclusion sequent Σ-calculus $\mathcal{G}$, we have the [finite] Hilbert-style Σ-calculus $(\mathcal{G}{\downarrow})\triangleq\{(\mathrm{img}\,\Gamma)|\varphi\mid(\Gamma\vdash\varphi)\in\mathcal{G}\}$. Conversely, given a Hilbert-style Σ-calculus $\mathcal{C}$, we have the axiomatic purely single-conclusion sequent Σ-calculus $(\mathcal{C}{\uparrow})\triangleq\{(\Gamma\vdash\varphi)\in\mathrm{Seq}^{2\backslash 1}_{\Sigma}\mid((\mathrm{img}\,\Gamma)|\varphi)\in\mathcal{C}\}$, in which case $(\mathcal{C}{\uparrow}{\downarrow})=\mathcal{C}$, as well as the Hilbert-style Σ-calculus $(\mathcal{C}{\Uparrow})\triangleq((\mathcal{C}\cap\mathrm{Fm}^{\omega}_{\Sigma})\cup(\sigma_{+1}[\mathcal{C}\backslash\mathrm{Fm}^{\omega}_{\Sigma}]\veebar x_0))$. Likewise, given an axiomatic [finite] multi-conclusion sequent Σ-calculus $\mathcal{G}$, we have the [finite] Hilbert-style Σ-calculi $(\mathcal{G}{\downarrow\downarrow})\triangleq(((\tau_{\veebar}[\mathcal{G}]\cap\mathrm{Seq}^{1\vdash(2\backslash 1)}_{\Sigma}){\downarrow})\cup(\sigma_{+1}[(\tau_{\veebar}[\mathcal{G}]\cap\mathrm{Seq}^{(\omega\backslash 1)\vdash(2\backslash 1)}_{\Sigma}\setminus\bigcup_{i\in\omega}(V^1_{\{i\}}\times(\mathrm{Fm}^{\omega\backslash\{i\}}_{\Sigma})^1)){\downarrow}]\veebar x_0)\cup\{(\sigma_{+1}[\mathrm{img}\,\Gamma]\veebar x_0)|x_0\mid\Gamma\in(\mathrm{Fm}^{\omega}_{\Sigma})^*,(\Gamma\vdash)\in\tau_{\veebar}[\mathcal{G}]\}\cup\{x_i|\varphi\mid i\in\omega,\varphi\in\mathrm{Fm}^{\omega\backslash\{i\}}_{\Sigma},(x_i\vdash\varphi)\in\tau_{\veebar}[\mathcal{G}]\})$ and $(\mathcal{G}{\Downarrow})\triangleq((\tau_{\veebar}[\mathcal{G}_{\backslash 1}]{\downarrow}){\Uparrow})$.

Lemma 5.10. *Let C be a $\veebar$-disjunctive Σ-logic and $\mathcal{S}$ an axiomatic multi-conclusion sequent Σ-calculus. Then, the extension of C relatively axiomatized by $\mathcal{S}{\downarrow\downarrow}$ is equally relatively axiomatized by $\mathcal{S}{\Downarrow}$, and so is $\veebar$-disjunctive.*

Proof. In that case, any extension of C satisfies (2.3), (2.5), (2.6) and (2.7), and so, given any $\Gamma\in\wp_{\omega}(\mathrm{Fm}^{\omega}_{\Sigma})$, applying the Σ-substitutions extending $[x_{i+1}/x_{i+2};x_0/(x_0\veebar x_1)]_{i\in\omega}$ and $[x_{i+2}/x_{i+1};x_j/x_0]^{j\in 2}_{i\in\omega}$ to the Σ-rules $(\sigma_{+1}[\Gamma]\veebar x_0)|x_0$ and $(\sigma_{+1}[\sigma_{+1}[\Gamma]]\veebar x_0)|(x_1\veebar x_0)$, respectively, we see that any extension of C satisfies the former iff it does the latter. Likewise, given any $i\in\omega$ and any $\varphi\in\mathrm{Fm}^{\omega\backslash\{i\}}_{\Sigma}$, applying the Σ-substitutions extending $[x_j/x_{j+1};x_i/(x_{i+1}\veebar x_0)]_{j\in(\omega\backslash\{i\})}$ and $[x_{j+1}/x_j;x_0/\varphi]_{j\in\omega}$ to the Σ-rules $x_i|\varphi$ and $(x_{i+1}\veebar x_0)|(\sigma_{+1}(\varphi)\veebar x_0)$, respectively, we see that any extension of C satisfies the former iff it does the latter. Finally, Corollary 3.5 completes the argument. □

Finally, given any (finite) $\mathcal{S}\subseteq\mathrm{Seq}^{\omega}_{\Sigma}$, we have the (finite) Σ-calculi $\mathcal{H}_{[1]}(\mathcal{M},\mathcal{S})\triangleq(\mathcal{D}_{\veebar}\cup((\mathcal{S}\cup\mathcal{A}_{[1]}(\mathcal{M})){\downarrow\downarrow}))$ and $\mathcal{H}'_{[1]}(\mathcal{M},\mathcal{S})\triangleq(\mathcal{D}_{\veebar}\cup((\mathcal{S}\cup\mathcal{A}_{[1]}(\mathcal{M})){\Downarrow}))$ (whenever Σ is finite, for $\mathcal{A}(\mathcal{M})$ is so, in that case).

Lemma 5.11. *Let C be a finitary $\veebar$-disjunctive Σ-logic and $\mathcal{S}$ an axiomatic purely single-conclusion sequent Σ-calculus. Suppose $(\mathcal{S}{\downarrow})\subseteq C$. Then, $C{\uparrow}$ is $(\mathcal{G}^{2\backslash 1}_{\veebar}\cup\mathcal{S})$-closed. In particular, $(\mathrm{Cn}_{\mathcal{G}^{2\backslash 1}_{\veebar}\cup\mathcal{S}}(\varnothing){\downarrow})\subseteq C$.*

Proof. Immediate, by the structurality and $\veebar$-disjunctivity of C. □

Lemma 5.12. *Let C be a $\veebar$-disjunctive Σ-logic. Then, any Σ-rule $(Y|\psi)$ is satisfied in C iff $\sigma_{+1}(Y|\psi)\veebar x_0$ is so.*

Proof. The "only if" part is by the structurality of C and Lemma 2.1(i)⇒(iii)(2.1) with $X = \varnothing$ and $a = x_0$. Conversely, assume $\sigma_{+1}(Y|\psi) \veebar x_0$ is satisfied in C, in which case $\sigma_{+1}(Y|\psi)$ is so, by Lemma 4.1, and so, applying the Σ-substitution extending $[x_{i+1}/x_i]_{i\in\omega}$, by the structurality of C, we eventually get $\psi \in C(Y)$, as required. □

Theorem 5.13. *Let $\mathcal{S} \subseteq \mathrm{Seq}^{\omega}_{\Sigma}$. Then, the logic of $\mathbf{S}_*(\mathsf{A}) \cap \mathrm{Mod}(\mathcal{S})$ is axiomatized by $\mathcal{H}_{[1]}(\mathcal{M}, \mathcal{S})$ or, equivalently, by $\mathcal{H}'_{[1]}(\mathcal{M}, \mathcal{S})$.*

Proof. First of all, recall that $\mathrm{Cn}_{\mathcal{D}_{\veebar}}$ is $\veebar$-disjunctive (cf. Theorem 4.4), and so is $C \triangleq \mathrm{Cn}_{\mathcal{H}_{[1]}(\mathcal{M},\mathcal{S})} = \mathrm{Cn}_{\mathcal{H}'_{[1]}(\mathcal{M},\mathcal{S})}$, in view of Lemma 5.10.

Next, every member of $\mathbf{S}_*(\mathsf{A}) \cap \mathrm{Mod}(\mathcal{S})$, being a $\veebar$-disjunctive model of $(\mathcal{S} \cup \mathcal{A}_{[1]}(\mathcal{M}))_{\backslash 1}$ (cf. Lemma 5.7), is so of $\tau_{\veebar}[(\mathcal{S} \cup \mathcal{A}_{[1]}(\mathcal{M}))_{\backslash 1}]$, and so of $\mathcal{H}'_{[1]}(\mathcal{M}, \mathcal{S})$, in view of Lemma 3.2.

Conversely, consider any Σ-rule $\mathcal{R} = (X|\phi)$ true in $\mathbf{S}_*(\mathsf{A}) \cap \mathrm{Mod}(\mathcal{S})$. Take any enumeration Γ of X. Then, the purely single-conclusion Σ-sequent $\Phi \triangleq (\Gamma \vdash \phi)$ is true in $\mathbf{S}(\mathsf{A}) \cap \mathrm{Mod}(\mathcal{S})$, and so is derivable in $\mathcal{G}^{2\backslash 1}_{\veebar} \cup \tau_{\veebar}[(\mathcal{S} \cup \mathcal{A}_{[1]}(\mathcal{M}))_{\backslash 1}]$, in view of Corollary 5.9. Moreover, by Lemma 5.12, we have $(\tau_{\veebar}[(\mathcal{S} \cup \mathcal{A}_{[1]}(\mathcal{M}))_{\backslash 1}]{\downarrow}) \subseteq C$. Hence, by Lemma 5.11, we eventually conclude that $\{\mathcal{R}\} = (\{\Phi\}{\downarrow}) \subseteq C$, as required. □

5.1. Disjunctive extensions

Lemma 5.14. $(\mathbf{S}(\mathcal{A}) \cap \mathrm{Mod}(\mathcal{A}'''_{[1]}(\mathcal{M}))) = \mathbf{S}(\mathsf{A})$.

Proof. The inclusion from right to left is by the inclusion $\mathsf{A} \subseteq \mathrm{Mod}(\mathcal{A}'''_{[1]}(\mathcal{M}))$ and the fact that the model classes of universal first-order theories are hereditary, while $\mathbf{S}$ is transitive. Conversely, consider any $\mathcal{D} \in (\mathbf{S}(\mathcal{A}) \setminus \mathbf{S}(\mathsf{A}))$. Then, for each $j \in m$, $D \nsubseteq B_j$, in which case there is some $d_j \in (D \setminus B_j) \neq \varnothing$. Let $(X/Y)_j \triangleq \{v \in \Upsilon \mid v^{\mathfrak{A}}(d_j) \in / \notin D^{\mathcal{A}}\}$, in which case $(X_j \cap Y_j) = \varnothing$, and $(k/l)_j \triangleq |(X/Y)_j| \in \omega$, respectively. Take any isomorphism $(\Gamma/\Delta)_j$ from $\langle (k/l)_j, \subseteq \cap (k/l)^2_j \rangle$ onto $\langle (X/Y)_j, \lesssim \cap (X/Y)^2_j \rangle$, respectively. Then, $\Omega_j \triangleq (\Gamma_j \vdash \Delta_j) \in \mathrm{Ax}(\Upsilon)$ is not true in $\mathcal{D}$ under $[x_0/d_j]$ but is true in $\mathcal{B}_j$, because $d_j \neq b$, for all $b \in B_j$, while Υ is an equality determinant for $\mathcal{A}$, in which case $\Omega_j \in \mathrm{Ax}(\mathcal{B}_j)$, while $\amalg \langle \Omega_j \rangle_{j\in m}$ is not true in $\mathcal{D}$ under $[x_j/d_j]_{j\in m}$, and so Claim 5.5 completes the argument. □

Lemma 5.15. *Let C be a Σ-logic and M a finite class of finite Σ-matrices. Suppose $C_{\dashv}$ is defined by M. Then, C is so too, that is, it is finitary.*

Proof. In that case, $\equiv_C = \equiv_{C_{\dashv}}$, while $C \supseteq C_{\dashv} = \mathrm{Cn}^{\omega}_{\mathsf{M}}$. To prove the converse is to prove that $\mathsf{M} \subseteq \mathrm{Mod}(C)$. For consider any $\mathcal{A} \in \mathsf{M}$, any $\Gamma \subseteq \mathrm{Fm}^{\omega}_{\Sigma}$, any $\varphi \in C(\Gamma)$ and any $h \in \hom(\mathfrak{Fm}^{\omega}_{\Sigma}, \mathfrak{A})$ such that $h[\Gamma] \subseteq D^{\mathcal{A}}$. Then, $\alpha \triangleq |A| \in (\omega \setminus 1) \subseteq \wp_{\infty\backslash 1}(\omega)$. Take any bijection $e : V_{\alpha} \to A$ to be extended to a $g \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{A})$. Then, $e^{-1} \circ (h{\restriction}V_{\omega})$ is extended to a Σ-substitution σ, in which case $\sigma(\varphi) \in C(\sigma[\Gamma])$, for C is structural, while $\sigma[\Gamma \cup \{\varphi\}] \subseteq \mathrm{Fm}^{\alpha}_{\Sigma}$. For every $\mathcal{B} \in \mathsf{M}$, we have the equivalence relation $\theta^{\mathcal{B}} \triangleq \{\langle a, b\rangle \in B^2 \mid (a \in D^{\mathcal{B}}) \Leftrightarrow (b \in D^{\mathcal{B}})\}$ on B. Moreover, as both α, M and all members of it are finite, we have the finite set $I \triangleq \{\langle h', \mathcal{B}\rangle \mid \mathcal{B} \in \mathsf{M}, h' \in \hom(\mathfrak{Fm}^{\alpha}_{\Sigma}, \mathfrak{B})\}$, in which case, for each $i \in I$, we set $h_i \triangleq \pi_0(i)$, $\mathcal{B}_i \triangleq \pi_1(i)$ and $\theta_i \triangleq \theta^{\mathcal{B}_i}$. Then, by (3.1), we have $\theta \triangleq (\equiv_{C_{\dashv}} \cap (\mathrm{Fm}^{\alpha}_{\Sigma})^2) = ((\mathrm{Fm}^{\alpha}_{\Sigma})^2 \cap \bigcap_{i\in I} h_i^{-1}[\theta_i])$, in which case, for every $i \in I$, $\theta \subseteq h_i^{-1}[\theta_i] = \ker(\nu_{\theta_i} \circ h_i)$, and so $g_i \triangleq (\nu_{\theta_i} \circ h_i \circ \nu_{\theta}^{-1}) : (\mathrm{Fm}^{\alpha}_{\Sigma}/\theta) \to B_i$. In this way, $f : (\mathrm{Fm}^{\alpha}_{\Sigma}/\theta) \to (\prod_{i\in I} B_i), a \mapsto \langle g_i(a)\rangle_{i\in I}$ is injective, for $(\ker f) =$

$((\mathrm{Fm}^{\alpha}_{\Sigma}/\theta)^2 \cap \bigcap_{i\in I}(\ker g_i))$ is diagonal. Hence, $\mathrm{Fm}^{\alpha}_{\Sigma}/\theta$ is finite, for $\prod_{i\in I} B_i$ is so, and so is $(\sigma[\Gamma]/\theta) \subseteq (\mathrm{Fm}^{\alpha}_{\Sigma}/\theta)$. For each $c \in (\sigma[\Gamma]/\theta)$, choose any $\phi_c \in (\sigma[\Gamma]\cap\nu_\theta^{-1}[\{c\}]) \neq \varnothing$. Put $\Delta \triangleq \{\phi_c \mid c \in (\sigma[\Gamma]/\theta)\} \in \wp_\omega(\sigma[\Gamma])$. Consider any $\psi \in \sigma[\Gamma]$. Then, $\Delta \ni \phi_{\nu_\theta(\psi)} \equiv_C \psi$, in which case $\psi \in C(\Delta)$, and so $\sigma[\Gamma] \subseteq C(\Delta)$. In this way, $\sigma(\varphi) \in C(\Delta) = C_{\dashv}(\Delta)$, for $\Delta \in \wp_\omega(\mathrm{Fm}^{\omega}_{\Sigma})$, so, by (3.1), $\sigma(\varphi) \in \mathrm{Cn}^{\alpha}_{\mathsf{M}}(\Delta)$. Moreover, $g[\Delta] \subseteq g[\sigma[\Gamma]] = h[\Gamma] \subseteq D^{\mathcal{A}}$, and so $h(\varphi) = g(\sigma(\varphi)) \in D^{\mathcal{A}}$, as required. □

Theorem 5.16. *Let C be the logic of A. Then, the following hold:*

(i) *The mappings:*

$$C' \mapsto (\mathbf{S}_*(\mathsf{A}) \cap \mathrm{Mod}(\mathcal{S})),$$
$$\mathsf{C} \mapsto \mathrm{Cn}_{\mathsf{C}}$$

are inverse to one another dual isomorphisms between the posets of all $\veebar$-disjunctive extensions of C and of all consistently hereditary subsets of $\mathbf{S}_*(\mathsf{A})$;

(ii) *the latter poset forms a finite distributive lattice, and so does the former one;*

(iii) *given a Σ-calculus $\mathcal{C}$, the extension of C relatively axiomatized by $\mathcal{C}$, being $\veebar$-disjunctive, corresponds to the consistently hereditary subset of $\mathbf{S}_*(\mathsf{A})$ relatively axiomatized by $\mathcal{C}$;*

(iv) *given an axiomatic multi-conclusion sequent Σ-calculus $\mathcal{S}$, the consistently hereditary subset of $\mathbf{S}_*(\mathsf{A})$ relatively axiomatized by $\mathcal{S}$ corresponds to the $\veebar$-disjunctive extension of C relatively axiomatized by $\mathcal{S}(\downdownarrows/\Downarrow)$;*

(v) *given any $\mathsf{K} \subseteq \mathbf{S}_*(\mathsf{A})$ and any MCES $\mathcal{N}$ for it, the logic of K is the $\veebar$-disjunctive extension of C relatively axiomatized by $\mathcal{A}'''_{[1]}(\mathcal{N})(\downdownarrows/\Downarrow)$ and corresponding to $\mathbf{S}_*(\mathsf{K})$.*

In particular, any $\veebar$-disjunctive extension of C is finitary and finitely-relatively-axiomatizable.

Proof. First, consider any $\veebar$-disjunctive extension C' of C, in which case, by Lemma 2.1(i)⇔(iii), $C'_{\dashv}$ is a finitary $\veebar$-disjunctive extension of C, and so is axiomatized by a Σ-calculus $\mathcal{C}$ (in particular, relatively to C) — e.g., by $C'_{\dashv}$ itself. Then, $\mathsf{C} \triangleq (\mathbf{S}_*(\mathsf{A}) \cap \mathrm{Mod}(C'_{\dashv})) = (\mathbf{S}_*(\mathsf{A}) \cap \mathrm{Mod}(\mathcal{C}))$ is the consistently hereditary subset of $\mathbf{S}_*(\mathsf{A})$ relatively axiomatized by $\mathcal{C}$. Clearly, $C'_{\dashv} \subseteq \mathrm{Cn}_{\mathsf{C}}$. Conversely, by Theorem 5.13 with $\mathcal{S} = (\mathcal{C}\uparrow)$, Cn_{C} is the extension of C relatively axiomatized by $\mathcal{C}\Uparrow$. On the other hand, as $\mathcal{C} \subseteq C'_{\dashv}$, by Lemma 5.12, we have $(\mathcal{C}\Uparrow) \subseteq C'_{\dashv}$. Thus, $C'_{\dashv} = \mathrm{Cn}_{\mathsf{C}}$, in which case, by Lemma 5.15, $C' = \mathrm{Cn}_{\mathsf{C}}$, and so $C' = C'_{\dashv}$.

Next, consider any $\mathsf{K} \subseteq \mathbf{S}_*(\mathsf{A})$. Then, by Lemma 3.2, Cn_{K} is a $\veebar$-disjunctive extension of C. Take any MCES $\mathcal{N}$ for K. Then, applying Theorem 5.13 with $\mathcal{S} = \varnothing$ twice (the second time — with $\mathcal{N}$ instead of $\mathcal{M}$), we see that Cn_{K} is axiomatized by $\mathcal{A}'''_{[1]}(\mathcal{N})(\downdownarrows/\Downarrow)$ relatively to C. In particular, by Lemmas 3.2, 5.7, 5.12, 5.14 and the $\veebar$-disjunctivity of submatrices of $\mathcal{A}$, we conclude that $(\mathbf{S}_*(\mathsf{A}) \cap \mathrm{Mod}(\mathrm{Cn}_{\mathsf{K}})) = (\mathbf{S}_*(\mathsf{A}) \cap \mathrm{Mod}(\mathcal{A}'''_{[1]}(\mathcal{N})\Downarrow)) = (\mathbf{S}_*(\mathsf{A}) \cap \mathrm{Mod}(\tau_{\veebar}[\mathcal{A}'''_{[1]}(\mathcal{N})_{\setminus 1}])) = (\mathbf{S}_*(\mathsf{A}) \cap \mathrm{Mod}(\mathcal{A}'''_{[1]}(\mathcal{N})) = \mathbf{S}_*(\mathsf{K})$, being equal to K, whenever this is consistently hereditary.

In this way, we have proved (i,iii,v), and so the final assertion. At last, (iv) is by Theorem 5.13, while (ii) is by the fact that the set of all consistently hereditary subsets of $\mathbf{S}_*(\mathsf{A})$ is a closure system over it closed under unions. □

As it is demonstrated in Subsubsection 6.2.4, $\mathcal{S}(\downdownarrows/\Downarrow)$/the reservation "being $\veebar$-disjunctive" cannot be, generally speaking, replaced with $\tau_{\veebar}[\mathcal{S}_{\setminus 1}]\downarrow$/omitted in the item (iv/iii) of Theorem 5.16, respectively.

5.2. Implicative case

Here, $\mathcal{A}$ is supposed to be a finite $\rhd$-implicative Σ-matrix with equality determinant $\Upsilon \ni x_0$, in which case it is $\veebar$-disjunctive, where $\veebar \triangleq \veebar_{\rhd}$ is *not* primary, and so is properly covered by the above discussion. Let $\theta_{\rhd} : \mathrm{Seq}_{\Sigma}^{2\backslash 1} \to \mathrm{Fm}_{\Sigma}^{\omega}, (\Gamma \vdash \phi) \mapsto (\Gamma \rhd \phi)$.

Example 5.17. When $\Upsilon \ni \upsilon = x_0$ and $\Sigma \ni F = \rhd$, in which case $\rhd$ is a primary binary connective of Σ, and so $\langle \upsilon, F\rangle$ is Υ-complex, one can always take $\lambda_{\mathcal{T}}(\upsilon, F) = \{\vdash x_0; x_1 \vdash\}$ and $\rho_{\mathcal{T}}(\upsilon, F) = \{x_0 \vdash x_1\}$ to satisfy (5.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(\upsilon, F) = \{x_0, (x_0 \rhd x_1) \vdash x_1\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(\upsilon, F) = \{\vdash x_0, (x_0 \rhd x_1); x_1 \vdash (x_0 \rhd x_1)\}$, and so elements of both $\theta_{\rhd}[\tau_{\veebar}[\boldsymbol{\lambda}_{\mathcal{T}}(\upsilon, F)]] = \{x_0 \rhd ((x_0 \rhd x_1) \rhd x_1)\}$ and $\theta_{\rhd}[\tau_{\veebar}[\boldsymbol{\rho}_{\mathcal{T}}(\upsilon, F)]] = \{(x_0 \rhd (x_0 \rhd x_1)) \rhd (x_0 \rhd x_1), (3.5)[x_0/x_1, x_1/x_0]\}$ are derivable in $\mathcal{I}_{\rhd}$, in view of Lemma 3.8, (3.2), (3.3) and (3.5). $\square$

In this way, let $\mathcal{A}'_{(\not\rhd)}$ be the set of all elements of $\boldsymbol{\lambda}_{\mathcal{T}}(\upsilon, F) \cup \boldsymbol{\rho}_{\mathcal{T}}(\upsilon, F)$, for all Υ-complex $\langle \Upsilon, \Sigma\rangle$-types $\langle \upsilon, F\rangle$ (but $\langle x_0, \rhd\rangle$, in case $\rhd \in \Sigma$ is primary). Then, set $\mathcal{A}_{[1](\not\rhd)}(\mathcal{M}) \triangleq (\mathcal{A}'_{(\not\rhd)} \cup \mathcal{A}'' \cup \mathcal{A}'''_{[1]}(\mathcal{M}))$ and $\mathcal{I}_{[1](\not\rhd)}(\mathcal{M}, \mathcal{S}) \triangleq (\mathcal{I}^{\mathrm{PL}}_{\rhd} \cup \theta_{\rhd}[\tau_{\veebar}[\mathcal{S}_{\backslash 1} \cup \mathcal{A}_{[1](\not\rhd)}(\mathcal{M})_{\backslash 1}]])$, where $\mathcal{S} \subseteq \mathrm{Seq}_{\Sigma}^{\omega}$.

Theorem 5.18. *Let $\mathcal{S} \subseteq \mathrm{Seq}_{\Sigma}^{\omega}$. Then, the logic of $\mathbf{S}_*(\mathsf{A}) \cap \mathrm{Mod}(\mathcal{S})$ is axiomatized by $\mathcal{I}_{[1]\not\rhd}(\mathcal{M}, \mathcal{S})$.*

Proof. First of all, note that $C \triangleq \mathrm{Cn}_{\mathcal{I}_{[1]\not\rhd}(\mathcal{M},\mathcal{S})}$ is equally axiomatized by $\mathcal{I}_{[1]}(\mathcal{M}, \mathcal{S})$, in view of Example 5.17, and is $\veebar$-disjunctive, by Theorem 3.9.

Next, every member of $\mathbf{S}_*(\mathsf{A}) \cap \mathrm{Mod}(\mathcal{S})$, being an $\rhd$-implicative (in particular, $\veebar$-disjunctive) model of $\mathcal{S}_{\backslash 1} \cup \mathcal{A}(\mathcal{M})_{\backslash 1}$ (cf. Lemma 5.7), is so of $\tau_{\veebar}[\mathcal{S}_{\backslash 1} \cup \mathcal{A}_{[1]}(\mathcal{M})_{\backslash 1}]$, and so of $\mathcal{I}_{[1]}(\mathcal{M}, \mathcal{S})$.

Conversely, consider any Σ-rule $\mathcal{R} = (X|\varphi)$ true in $\mathbf{S}_*(\mathsf{A}) \cap \mathrm{Mod}(\mathcal{S})$. Take any enumeration Γ of X. Then, the purely single-conclusion Σ-sequent $\Phi \triangleq (\Gamma \vdash \varphi)$ is true in $\mathbf{S}(\mathsf{A}) \cap \mathrm{Mod}(\mathcal{S})$, and so is derivable in $\mathcal{G}_{\veebar}^{2\backslash 1} \cup \tau_{\veebar}[\mathcal{S}_{\backslash 1} \cup \mathcal{A}_{[1]}(\mathcal{M})_{\backslash 1}]$, in view of Corollary 5.9. Finally, as $\theta_{\rhd}[\tau_{\veebar}[\mathcal{S}_{\backslash 1} \cup \mathcal{A}_{[1]}(\mathcal{M})_{\backslash 1}]] \subseteq \mathcal{I}(\mathcal{M}, \mathcal{S}) \subseteq C$, by (3.3), we have $(\tau_{\veebar}[\mathcal{S}_{\backslash 1} \cup \mathcal{A}_{[1]}(\mathcal{M})_{\backslash 1}]\downarrow) \subseteq C$, and so, by Lemma 5.11, we get $\{\mathcal{R}\} = (\{\Phi\}\downarrow) \subseteq C$. $\square$

Combining Theorems 5.16, 5.18 with Corollary 3.5, we have:

Corollary 5.19. *Let C be the logic of A and $\diamond$ any (possibly, secondary) binary connective of Σ. Suppose each member of A is $\diamond$-disjunctive. Then, any extension of C is $\veebar$-disjunctive iff it is $\diamond$-disjunctive iff it is axiomatic. Moreover, the following hold:*

(i) *given an axiomatic multi-conclusion sequent Σ-calculus $\mathcal{S}$, the consistently hereditary subset of $\mathbf{S}_*(\mathsf{A})$ relatively axiomatized by $\mathcal{S}$ defines the axiomatic extension of C relatively axiomatized by $\theta_{\rhd}[\tau_{\diamond}[\mathcal{S}_{\backslash 1}]]$;*

(ii) *given any $\mathsf{K} \in \mathbf{S}_*(\mathsf{A})$ and any MCES $\mathcal{N}$ for it, the logic of K is the axiomatic extension of C relatively axiomatized by $\theta_{\rhd}[\tau_{\diamond}[\mathcal{A}'''_{[1]}(\mathcal{N})_{\backslash 1}]]$.*

6. Applications and examples

Here, we use Theorems 5.13 and 5.18 tacitly, following notations adopted in the previous section and, unless otherwise specified, supposing that $\mathsf{A} = \{\mathcal{A}\}$, and so $\mathcal{M} = \{\langle 0, \mathsf{A}\rangle\}$, in which case $\mathcal{A}_{(1)}[''']$ stands for $\mathcal{A}_{(1)}[''](\mathcal{M})[= \mathrm{Ax}^*_{(1)}(\mathcal{A})]$, respectively.

6.1. Disjunctive and implicative positive fragments of the classical logic

Here, we deal with the signature $\Sigma^{(\supset)}_{+[,01]} \triangleq (\{\wedge, \vee\}[\cup\{\bot, \top\}](\cup\{\supset\}))$. By $\mathfrak{D}^{(\supset)}_{n[,01]}$, where $n(=2) \in (\omega\backslash 1)$, we denote the $\Sigma^{(\supset)}_{+[,01]}$-algebra such that $\mathfrak{D}^{(\supset)}_{n[,01]}\restriction\Sigma_{+[,01]}$ is the [bounded] distributive lattice given by the chain n ordered by ordinal inclusion (and $(i \supset^{\mathfrak{D}^{\supset}_{2[,01]}} j) \triangleq (\max(1-i, j)$, for all $i, j \in 2$). Then, the logic of the $\vee$-disjunctive (and $\supset$-implicative) $\mathcal{D}^{(\supset)}_{2[,01]} \triangleq \langle \mathfrak{D}^{(\supset)}_{2[,01]}, \{1\}\rangle$ with equality determinant $\Upsilon = \{x_0\}$ (cf. Example 1 of Pynko (2004)) is the $\Sigma^{(\supset)}_{+[,01]}$-fragment of the classical logic. Throughout the rest of this subsection, it is supposed that $\Sigma \subseteq \Sigma^{(\supset)}_{+,01}$ and $\mathcal{A} = (\mathcal{D}^{(\supset)}_{2,01}\restriction\Sigma)$, in which case $\mathcal{A}''' = \varnothing$.

First, in case $\Sigma = \{\supset\}$, both $\mathcal{A}'_{\not\supset}$ and $\mathcal{A}''$ are empty, and so is $\mathcal{A}_{\not\supset}$. In this way, we have the following well-known result:

Corollary 6.1. *The $\{\supset\}$-fragment of the classical logic is axiomatized by $\mathcal{I}^{\mathrm{PL}}_{\supset}$.*

Likewise, in case $\Sigma = \{\vee\}$, both $\mathcal{A}'$ and $\mathcal{A}''$ are empty, and so is $\mathcal{A}$. In this way, we get:

Corollary 6.2. *The $\{\vee\}$-fragment of the classical logic is axiomatized by $\mathcal{D}_{\vee}$.*

Next, let $\Sigma = \Sigma_+$. Then, $\mathcal{A}'' = \varnothing$, while one can take $\lambda_{\mathcal{T}}(x_0, \wedge) = \{x_0, x_1 \vdash\}$ and $\rho_{\mathcal{T}}(x_0, \wedge) = \{\vdash x_0; \vdash x_1\}$ to satisfy (5.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(x_0, \wedge) = \{(x_0 \wedge x_1) \vdash x_0; (x_0 \wedge x_1) \vdash x_1\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(x_0, \wedge) = \{x_0, x_1 \vdash (x_0 \wedge x_1)\}$, and so $\mathcal{A} = \mathcal{A}' = \{(x_0 \wedge x_1) \vdash x_0; (x_0 \wedge x_1) \vdash x_1; x_0, x_1 \vdash (x_0 \wedge x_1)\}$. Thus, we get:

Corollary 6.3. *The Σ_+-fragment of the classical logic is axiomatized by the calculus $\mathcal{PC}_+$ resulted from $\mathcal{D}_{\vee}$ by adding the following rules:*

$$C_1 \quad \frac{(x_1 \wedge x_2) \vee x_0}{x_1 \vee x_0} \qquad C_2 \quad \frac{(x_1 \wedge x_2) \vee x_0}{x_2 \vee x_0} \qquad C_3 \quad \frac{x_1 \vee x_0; x_2 \vee x_0}{(x_1 \wedge x_2) \vee x_0}$$

It is remarkable that the calculus $\mathcal{PC}_+$ consists of seven rules, while that which was found in Dyrda and Prucnal (1980) has nine rules. This demonstrates the practical applicability of our generic approach (more precisely, its factual ability to result in really "good" calculi to be enhanced a bit more by replacing appropriate pairs of rules/premises with single ones upon the basis of Corollary 4.5 and rules C_i, where $i \in (4 \setminus 1)$, whenever it is possible, to be done below tacitly — "on the fly").

Likewise, let $\Sigma = \Sigma^{\supset}_+$. Then, $\mathcal{A}'' = \varnothing$, and so, taking Corollary 3.10(ii) and Example 5.1 into account, we have the following well-known result:

Corollary 6.4. *The $\Sigma^{\supset}_+$-fragment of the classical logic is axiomatized by the calculus*

$\mathcal{PC}_{+}^{\supset}$ *resulted from* $\mathcal{I}_{\supset}^{\mathrm{PL}}$ *by adding the following axioms:*

$$(x_0 \wedge x_1) \supset x_i \qquad\qquad x_0 \supset (x_1 \supset (x_0 \wedge x_1))$$
$$x_i \supset (x_0 \vee x_1) \qquad\qquad (x_0 \supset x_2) \supset ((x_1 \supset x_2) \supset ((x_0 \vee x_1) \supset x_2))$$

where $i \in 2$*.*

Finally, let $\Sigma = \Sigma_{+,01}^{[\supset]}$, in which case $\mathcal{A}'$ is as above, while $\mathcal{A}'' = \{\vdash \top; \bot \vdash\}$, and so [taking Corollary 3.10(ii) into account] we get:

Corollary 6.5. *The* $\Sigma_{+,01}^{[\supset]}$*-fragment of the classical logic is axiomatized by the calculus* $\mathcal{PC}_{+,01}^{[\supset]}$ *resulted from* $\mathcal{PC}_{+}^{[\supset]}$ *by adding the following rules:*

$$\top \qquad\qquad \frac{\bot \vee x_0}{x_0}[\bot \supset x_0]$$

6.2. Miscellaneous four-valued expansions of Belnap's four-valued logic

Let $\Sigma_{\sim,+[,01]}^{(\supset)} \triangleq (\Sigma_{+[,01]}^{(\supset)} \cup \{\sim\})$, where $\sim$ (weak negation) is unary.

Here, it is supposed that $\Sigma \supseteq \Sigma_{\sim,+[,01]}$, $(\mathfrak{A}{\restriction}\Sigma_{\sim,+[,01]}) = \mathfrak{DM}_{4[,01]}$, where $(\mathfrak{DM}_{4[,01]}{\restriction}\Sigma_{+[,01]}) \triangleq \mathfrak{D}_{2[,01]}^{2}$, while $\sim^{\mathfrak{DM}_{4[,01]}}\langle i,j\rangle \triangleq \langle 1-j, 1-i\rangle$, for all $i,j \in 2$, in which case we use the following standard notations for elements of 2^2 going back to Belnap (1977):

$$\mathsf{t} \triangleq \langle 1,1\rangle, \qquad \mathsf{f} \triangleq \langle 0,0\rangle, \qquad \mathsf{b} \triangleq \langle 1,0\rangle, \qquad \mathsf{n} \triangleq \langle 0,1\rangle,$$

and $\mathcal{A} \triangleq \langle \mathfrak{A}, \{\mathsf{b},\mathsf{t}\}\rangle$, in which case it is $\vee$-disjunctive, while $\Upsilon = \{x_0, \sim x_0\}$ is an equality determinant for it (cf. Example 2 of Pynko (2004)), whereas $\mathcal{A}''' = \varnothing$. (Since the logic $B_{4[,01]}$ of $\mathcal{DM}_{4[,01]} \triangleq (\mathcal{A}{\restriction}\Sigma_{\sim,+[,01]})$ is the [bounded version of] Belnap's logic, the logic of $\mathcal{A}$ is a four-valued expansion of $B_{4[,01]}$.)

First, let $\Sigma = \Sigma_{\sim,+}$, in which case $\mathcal{A}'' = \varnothing$, while the case of the Υ-complex $\langle\Upsilon,\Sigma\rangle$-type $\langle x_0, \wedge\rangle$ is as in the previous subsection, whereas others but $\langle x_0, \vee\rangle$ are as follows. First of all, one can take $\lambda_{\mathcal{T}}(\sim x_0, \vee) = \{\sim x_0, \sim x_1 \vdash\}$ and $\rho_{\mathcal{T}}(\sim x_0, \vee) = \{\vdash \sim x_0; \vdash \sim x_1\}$ to satisfy (5.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(\sim x_0, \vee) = \{\sim(x_0 \vee x_1) \vdash \sim x_0; \sim(x_0 \vee x_1) \vdash \sim x_1\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(\sim x_0, \vee) = \{\sim x_0, \sim x_1 \vdash \sim(x_0 \vee x_1)\}$. Likewise, one can take $\lambda_{\mathcal{T}}(\sim x_0, \wedge) = \{\sim x_0 \vdash; \sim x_1 \vdash\}$ and $\rho_{\mathcal{T}}(\sim x_0, \wedge) = \{\vdash \sim x_0, \sim x_1\}$ to satisfy (5.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(\sim x_0, \wedge) = \{\sim(x_0 \wedge x_1) \vdash \sim x_0, \sim x_1\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(\sim x_0, \wedge) = \{\sim x_0 \vdash \sim(x_0 \wedge x_1); \sim x_1 \vdash \sim(x_0 \wedge x_1)\}$. Finally, one can take $\lambda_{\mathcal{T}}(\sim x_0, \sim) = \{x_0 \vdash\}$ and $\rho_{\mathcal{T}}(\sim x_0, \sim) = \{\vdash x_0\}$ to satisfy (5.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(\sim x_0, \sim) = \{\sim\sim x_0 \vdash x_0\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(\sim x_0, \sim) = \{x_0 \vdash \sim\sim x_0\}$. In this way, we get:

Corollary 6.6. B_4 *is axiomatized by the calculus* $\mathcal{B}$ *resulted from* $\mathcal{PC}_{+}$ *by adding the following rules as well as the inverse to these:*

$$\overset{NN}{\frac{x_1 \vee x_0}{\sim\sim x_1 \vee x_0}} \qquad \overset{ND}{\frac{(\sim x_1 \wedge \sim x_2) \vee x_0}{\sim(x_1 \vee x_2) \vee x_0}} \qquad \overset{NC}{\frac{(\sim x_1 \vee \sim x_2) \vee x_0}{\sim(x_1 \wedge x_2) \vee x_0}}$$

The calculus $\mathcal{B}$ has 13 rules, while the very first axiomatization of B_4 discovered in

Pynko (1995a) (cf. Definition 5.1 and Theorem 5.2 therein)[2] has 15 rules, "two rules win" being just due to the advance of the present study with regard to Dyrda and Prucnal (1980) (cf. the previous subsection).

Now, let $\Sigma = \Sigma_{\sim,+,01}$, in which case both $\mathcal{A}'$ and $\mathcal{A}'''$ are as above, while $\mathcal{A}'' = \{\top; \sim\bot; \bot \vdash; \sim\top \vdash\}$, and so we get:

Corollary 6.7. *$B_{4,01}$ is axiomatized by the calculus $\mathcal{B}_{01}$ resulted from $\mathcal{B} \cup \mathcal{PC}_{+,01}$ by adding the following axiom and rule:*

$$\sim\bot \qquad\qquad \frac{\sim\top \vee x_0}{x_0}$$

6.2.1. The classical expansion

Let $\Sigma^{(\supset)}_{\simeq,+[,01]} \triangleq (\Sigma^{(\supset)}_{\sim,+[,01]} \cup \{\neg\})$, where $\neg$ (classical negation) is unary.

Here, it is supposed that $\Sigma = \Sigma_{\simeq,+[,01]}$, while $\neg^{\mathfrak{A}}\langle i,j\rangle \triangleq \langle 1-i, 1-j\rangle$, for all $i,j \in 2$. Then, one can take $\lambda_{\mathcal{T}}(x_0,\neg) = \{\vdash x_0\}$ and $\rho_{\mathcal{T}}(x_0,\neg) = \{x_0 \vdash\}$ to satisfy (5.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(x_0,\neg) = \{x_0, \neg x_0 \vdash\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(x_0,\neg) = \{\vdash x_0, \neg x_0\}$. Likewise, one can take $\lambda_{\mathcal{T}}(\sim x_0,\neg) = \{\vdash \sim x_0\}$ and $\rho_{\mathcal{T}}(\sim x_0,\neg) = \{\sim x_0 \vdash\}$ to satisfy (5.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(\sim x_0,\neg) = \{\sim x_0, \sim\neg x_0 \vdash\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(\sim x_0,\neg) = \{\vdash \sim x_0, \sim\neg x_0\}$. Thus, we get:

Corollary 6.8. *The logic of $\mathcal{A}$ is axiomatized by the calculus $\mathcal{CB}_{[01]}$ resulted from $\mathcal{B}_{[01]}$ by adding the following rules:*

$$\overset{N_1}{\frac{(x_1 \wedge \neg x_1) \vee x_0}{x_0}} \qquad \overset{N_2}{x_0 \vee \neg x_0} \qquad \overset{N_3}{\frac{(\sim x_1 \wedge \sim\neg x_1) \vee x_0}{x_0}} \qquad \overset{N_4}{\sim x_0 \vee \sim\neg x_0}$$

6.2.2. The bilattice expansions

Let $\Sigma^{(\supset)}_{\sim/\simeq,2:+[,01]} \triangleq (\Sigma^{(\supset)}_{\sim/\simeq,+[,01]} \cup \{\sqcap,\sqcup\}[\cup\{\mathbf{0},\mathbf{1}\}])$, where $\sqcap$ and $\sqcup$ (*knowledge* conjunction and disjunction, respectively) are binary [while $\mathbf{0}$ and $\mathbf{1}$ are nullary].

Here, it is supposed that $\Sigma = \Sigma_{\sim/\simeq,2:+[,01]}$, while

$$(\langle i,j\rangle(\sqcap/\sqcup)^{\mathfrak{A}}\langle k,l\rangle) \triangleq \langle(\min/\max)(i,k), (\max/\min)(j,l)\rangle,$$

for all $i,j,k,l \in 2$ [whereas $\mathbf{0}^{\mathfrak{A}} \triangleq \mathsf{n}$ and $\mathbf{1}^{\mathfrak{A}} \triangleq \mathsf{b}$].

First, let $\Sigma = \Sigma_{\sim,2:+}$, in which case $\mathcal{A}'' = \varnothing$. Then, one can take $\lambda_{\mathcal{T}}(x_0,\sqcap) = \{x_0, x_1 \vdash\}$ and $\rho_{\mathcal{T}}(x_0,\sqcap) = \{\vdash x_0; \vdash x_1\}$ to satisfy (5.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(x_0,\sqcap) = \{(x_0 \sqcap x_1) \vdash x_0; (x_0 \sqcap x_1) \vdash x_1\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(x_0,\sqcap) = \{x_0, x_1 \vdash (x_0 \sqcap x_1)\}$. Likewise, one can take $\lambda_{\mathcal{T}}(x_0,\sqcup) = \{x_0 \vdash; x_1 \vdash\}$ and $\rho_{\mathcal{T}}(x_0,\sqcup) = \{\vdash x_0, x_1\}$ to satisfy (5.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(x_0,\sqcup) = \{(x_0 \sqcup x_1) \vdash x_0, x_1\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(x_0,\sqcup) = \{x_0 \vdash (x_0 \sqcup x_1); x_1 \vdash (x_0 \sqcup x_1)\}$. Next, one can take $\lambda_{\mathcal{T}}(\sim x_0,\sqcap) = \{\sim x_0, \sim x_1 \vdash\}$ and $\rho_{\mathcal{T}}(\sim x_0,\sqcap) = \{\vdash \sim x_0; \vdash \sim x_1\}$ to satisfy (5.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(\sim x_0,\sqcap) = \{\sim(x_0 \sqcap x_1) \vdash \sim x_0; \sim(x_0 \sqcap x_1) \vdash \sim x_1\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(\sim x_0,\sqcap) = \{\sim x_0, \sim x_1 \vdash \sim(x_0 \sqcap x_1)\}$. Finally, one can take $\lambda_{\mathcal{T}}(\sim x_0,\sqcup) = \{\sim x_0 \vdash; \sim x_1 \vdash\}$ and $\rho_{\mathcal{T}}(\sim x_0,\sqcup) = \{\vdash \sim x_0, \sim x_1\}$ to satisfy (5.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(\sim x_0,\sqcup) =$

[2] In this connection, we should like to take the opportunity to specify the ambiguous footnote 3 on p. 443 therein. The problem has been that, as we have noticed, because of missing a reservation like "in reply to our first informing him about this result about two weeks before" just after "1994", the mentioned footnote has been misleading readers leaving them with wrong impression about the genuine priority/authorship as to this result.

$\{\sim(x_0 \sqcup x_1) \vdash \sim x_0, \sim x_1\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(\sim x_0, \sqcup) = \{\sim x_0 \vdash \sim(x_0 \sqcup \sim x_1); \sim x_1 \vdash \sim(x_0 \sqcup x_1)\}$. Thus, we get:

Corollary 6.9. *The logic of $\mathcal{A}$ is axiomatized by the calculus $\mathcal{BL}$ resulted from adding to $\mathcal{B}$ the following rules as well as the inverse to these:*

$$\begin{array}{cccc} KC & KD & NKC & NKD \\ \dfrac{(x_1 \wedge x_2) \vee x_0}{(x_1 \sqcap x_2) \vee x_0} & \dfrac{(x_1 \vee x_2) \vee x_0}{(x_1 \sqcup x_2) \vee x_0} & \dfrac{(\sim x_1 \wedge \sim x_2) \vee x_0}{\sim(x_1 \sqcap x_2) \vee x_0} & \dfrac{(\sim x_1 \vee \sim x_2) \vee x_0}{\sim(x_1 \sqcup x_2) \vee x_0} \end{array}$$

Likewise, let $\Sigma = \Sigma_{\sim,2:+,01}$, in which case both $\mathcal{A}'$ and $\mathcal{A}'''$ are as above, while $\mathcal{A}'' = (\{\bot \vdash; \top\} \cup \{\sim^i \mathbf{0} \vdash; \sim^i \mathbf{1} \mid i \in 2\})$, and so we have:

Corollary 6.10. *The logic of $\mathcal{A}$ is axiomatized by the calculus $\mathcal{BL}_{01}$ resulted from adding to $\mathcal{BL} \cup \mathcal{B}_{01}$ the following axioms and rules:*

$$\sim^i \mathbf{1} \qquad\qquad \frac{\sim^i \mathbf{0} \vee x_0}{x_0}$$

where $i \in 2$.

Finally, when $\Sigma = \Sigma_{\simeq,2:+[,01]}$, we have:

Corollary 6.11. *The logic of $\mathcal{A}$ is axiomatized by the calculus $\mathcal{CB} \cup \mathcal{BL}_{[01]}$.*

6.2.3. Implicative expansions

Here, it is supposed that $\supset \in \Sigma$, while $(\langle i, j\rangle \supset^{\mathfrak{A}} \langle k, l\rangle) \triangleq \langle \max(1-i, k), \max(1-i, l)\rangle$, for all $i, j, k, l \in 2$, in which case $\mathcal{A}$ is $\supset$-implicative.

First, let $\Sigma = \Sigma^{\supset}_{\sim,+}$. Clearly, one can take $\lambda_{\mathcal{T}}(\sim x_0, \supset) = \{x_0, \sim x_1 \vdash\}$ and $\rho_{\mathcal{T}}(\sim x_0, \supset) = \{\vdash x_0; \vdash \sim x_1\}$ to satisfy (5.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(\sim x_0, \supset) = \{\sim(x_0 \supset x_1) \vdash x_0; \sim(x_0 \supset x_1) \vdash \sim x_1\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(\sim x_0, \supset) = \{x_0, \sim x_1 \vdash \sim(x_0 \supset x_1)\}$. Therefore, taking Corollary 3.10(ii) and Example 5.1 into account, we get:

Corollary 6.12. *The logic of $\mathcal{A}$ is axiomatized by the calculus $\mathcal{B}^{\supset}$ resulted from $\mathcal{PC}^{\supset}_{+}$ by adding the following axioms:*

$$\sim\sim x_0 \supset x_0 \qquad x_0 \supset \sim\sim x_0 \tag{6.1}$$

$$\sim(x_0 \vee x_1) \supset \sim x_i \qquad \sim x_0 \supset (\sim x_1 \supset \sim(x_0 \vee x_1)) \tag{6.2}$$

$$\sim x_i \supset \sim(x_0 \wedge x_1) \qquad (\sim x_0 \supset x_2) \supset ((\sim x_1 \supset x_2) \supset (\sim(x_0 \wedge x_1) \supset x_2)) \tag{6.3}$$

$$\sim(x_0 \supset x_1) \supset \sim^i x_i \qquad x_0 \supset (\sim x_1 \supset \sim(x_0 \supset x_1))$$

where $i \in 2$.

It is remarkable that $\mathcal{B}^{\supset}$ is actually the calculus *Par* introduced in Popov (1989) but regardless to any semantics. In this way, the present study provides a new (and quite immediate) insight into the issue of semantics of *Par* first being due to Pynko (1999) but with using the intermediate purely multi-conclusion sequent calculus *GPar* actually introduced in Popov (1989) regardless to any semantics too and then studied semantically in Pynko (1999).

Likewise, in case $\Sigma = \Sigma^{\supset}_{\sim,+,01}$, we have:

Corollary 6.13. *The logic of* $\mathcal{A}$ *is axiomatized by the calculus* $\mathcal{B}^{\supset}_{01}$ *resulted from* $\mathcal{B}^{\supset} \cup \mathcal{PC}^{\supset}_{+,01}$ *by adding the following axioms:*

$$\sim\bot \qquad\qquad \sim\top \supset x_0$$

Now, let $\Sigma = \Sigma^{\supset}_{\sim,2:+}$. Then, we have:

Corollary 6.14. *The logic of* $\mathcal{A}$ *is axiomatized by the calculus* $\mathcal{BL}^{\supset}$ *resulted from* $\mathcal{B}^{\supset}$ *by adding the following axioms:*

$$(x_0 \sqcap x_1) \supset x_i \qquad x_0 \supset (x_1 \supset (x_0 \sqcap x_1))$$
$$x_i \supset (x_0 \sqcup x_1) \qquad (x_0 \supset x_2) \supset ((x_1 \supset x_2) \supset ((x_0 \sqcup x_1) \supset x_2))$$
$$\sim(x_0 \sqcap x_1) \supset \sim x_i \qquad \sim x_0 \supset (\sim x_1 \supset \sim(x_0 \sqcap x_1))$$
$$\sim x_i \supset \sim(x_0 \sqcup x_1) \qquad (\sim x_0 \supset x_2) \supset ((\sim x_1 \supset x_2) \supset (\sim(x_0 \sqcup x_1) \supset x_2))$$

where $i \in 2$.

Likewise, when $\Sigma = \Sigma^{\supset}_{\sim,2:+,01}$, we have:

Corollary 6.15. *The logic of* $\mathcal{A}$ *is axiomatized by the calculus* $\mathcal{BL}^{\supset}_{01}$ *resulted from* $\mathcal{BL}^{\supset} \cup \mathcal{B}^{\supset}_{01}$ *by adding the following axioms:*

$$\sim^i \mathbf{1} \qquad\qquad \sim^i \mathbf{0} \supset x_0$$

where $i \in 2$.

Further, let $\Sigma = \Sigma^{\supset}_{\simeq,+[,01]}$. Then, taking (3.3) and Corollary (3.10)(i) into account, we have:

Corollary 6.16. *The logic of* $\mathcal{A}$ *is axiomatized by the calculus* $\mathcal{CB}^{\supset}_{[01]}$ *resulted from* $\mathcal{B}^{\supset}_{[01]}$ *by adding the axioms* N_2, N_4 *and the following ones:*

$$\sim^i x_1 \supset (\sim^i \neg x_i \supset x_0),$$

where $i \in 2$.

Finally, when $\Sigma = \Sigma^{\supset}_{\simeq,2:+[,01]}$, we have:

Corollary 6.17. *The logic of* $\mathcal{A}$ *is axiomatized by the calculus* $\mathcal{CB}^{\supset} \cup \mathcal{BL}^{\supset}_{[01]}$.

6.2.4. Disjunctive extensions

Clearly, $(\mathbf{S}_*(\mathcal{A})[\setminus\{\{\mathsf{n}\}\}]) \subseteq \mathsf{S} \triangleq \mathbf{S}_*(\mathcal{DM}_{4[,01]}) = (\{A, A_{\not{\mathsf{n}}}, A_{\not{\mathsf{b}}}, A_{\not{\mathsf{n}}\not{\mathsf{b}}}, \{\mathsf{n}\}\}[\setminus\{\{\mathsf{n}\}\}])$, where $A_{(\not{\mathsf{n}})[\not{\mathsf{b}}]} \triangleq (A(\setminus\{\mathsf{n}\})[\setminus\{\mathsf{b}\}])$.

Remark 6.18. The mappings $\mathsf{C} \mapsto (\mathsf{S} \cap \bigcup\{\wp(S) \mid S \in \mathsf{C}\})$ and $\mathsf{C}' \mapsto (\mathsf{C}' \cap \mathbf{S}_*(\mathcal{A}))$ form a dual Galois retraction between the posets of all consistently hereditary subsets of $\mathbf{S}_*(\mathcal{A})$ and those of S, the former/latter one preserving generating subsets/relative axiomatizations, respectively. □

First of all, we analyze $\rho_{\mathsf{S}} \upharpoonright \mathrm{Ax}(\Upsilon)$:

$$
\begin{aligned}
\rho_{\mathsf{S}}(\vdash) &= \varnothing, \\
\rho_{\mathsf{S}}(\sim x_0, x_0 \vdash) &= \{\{\mathsf{n}\}, A_{\not\mathsf{b}}, A_{\not\mathsf{n}\not\mathsf{b}}\}, \\
\rho_{\mathsf{S}}(\vdash \sim x_0, x_0) &= \{A_{\not\mathsf{n}}, A_{\not\mathsf{n}\not\mathsf{b}}\}, \\
\rho_{\mathsf{S}}(\sim^i x_0 \vdash [\sim^{1-i} x_0]) &= \{\{\mathsf{n}\}\},
\end{aligned}
$$

where $i \in 2$. Moreover, since consistently hereditary proper subsets of the finite set S are generated by subsets of $\mathsf{S} \setminus \{\mathcal{A}\}$ being anti-chains, it suffices to to count $\mathsf{A} \subseteq \mathsf{S}$ an anti-chain not containing $\mathcal{A}$. On the other hand, these are either $\varnothing$ or singletons but $\{\mathcal{A}\}$ or $\{A_{\not\mathsf{n}[\not\mathsf{b}]}, \{\mathsf{n}\}\}$. Then, for every $\mathsf{X} \subseteq \mathsf{A}$, $\mathrm{Ax}(\Upsilon) \cap \rho_{\mathsf{A}}^{-1}[\{\mathsf{X}\}]$ is either empty or a singleton, and so is $\min_{\sqsubseteq}(\mathrm{Ax}(\Upsilon) \cap \rho_{\mathsf{A}}^{-1}[\{\mathsf{X}\}]) = (\mathrm{Ax}(\Upsilon) \cap \rho_{\mathsf{A}}^{-1}[\{\mathsf{X}\}])$, unless $\mathsf{X} = \{\{\mathsf{n}\}\}$, in which case $\min_{\sqsubseteq}(\mathrm{Ax}(\Upsilon) \cap \rho_{\mathsf{A}}^{-1}[\{\mathsf{X}\}]) = \{x_0 \vdash\}$.

By C^{EM} we denote the axiomatic extension of C relatively axiomatized by the *Excluded Middle Law* axiom:

$$x_0 \vee \sim x_0. \tag{6.4}$$

Then, $C^{[\mathrm{EM}+]\mathrm{R}}$ denotes the extension of $C^{[\mathrm{EM}]}$ relatively axiomatized by the *Resolution* rule:

$$\{x_1 \vee x_0, \sim x_1 \vee x_0\} | x_0. \tag{6.5}$$

In this way, choosing those enumerations of two-element anti-chains comprehensively described above, the first components of which contain $\{\mathsf{n}\}$, and combining Remarks 3.1, 6.18 and Proposition 3.6 with the "$\downarrow\downarrow$"-version of Theorem 5.16, we eventually get:

Theorem 6.19. *Suppose C has no/a theorem. Then, $\vee$-disjunctive extensions of C form a Galois retract of the (9/6)-element non-chain distributive lattice of all those of $B_{4/,01}$, depicted at Figure 1/ with solely solid circles, and are defined by all those consistent submatrices of $\mathcal{A}$, the carriers of whose underlying algebras are subsets of any element of those sets, which mark corresponding nodes, in which case different nodes may correspond to a same extension with different relative axiomatizations. Moreover, those of them, whose relative axiomatizations are not given by upper indices, are axiomatized relatively to C by the following calculi:*

$$
\begin{aligned}
C^{\mathrm{EM}} \cap C^{\mathrm{R}} &: \{x_1 \vee x_0, \sim x_1 \vee x_0\} \vdash ((x_2 \vee \sim x_2) \vee x_0), \\
\mathrm{IC} &: x_0, \\
\mathrm{IC}_{+0} &: x_1 | x_0, \\
C^{\mathrm{EM}}_{+0} &: x_0 | (x_1 \vee \sim x_1), \qquad\qquad (6.6) \\
C^{\mathrm{EM+R}}_{+0} &: \{(6.5), (6.6)\}.
\end{aligned}
$$

In case $A_{\not\mathsf{n}/\not\mathsf{b}}$ forms a subalgebra of $\mathfrak{A}$, $C^{\mathrm{EM/R}}$ thus covers arbitrary three-valued expansions of the *logic of paradox LP* Priest (1979) (cf. Pynko (1995b)) {including *LP* itself, when $\Sigma = \Sigma_{\sim,+}$, and so subsuming Corollary 5.3 of Pynko (1995a), its bounded expansion, when $\Sigma = \Sigma_{\sim,+,01}$, the *logic of antinomies LA* Asenjo and Tamburino (1975), when $\Sigma = \Sigma^{\supset}_{\sim,+}$}/*Kleene's three-valued logic* Kleene (1952). In particular, it

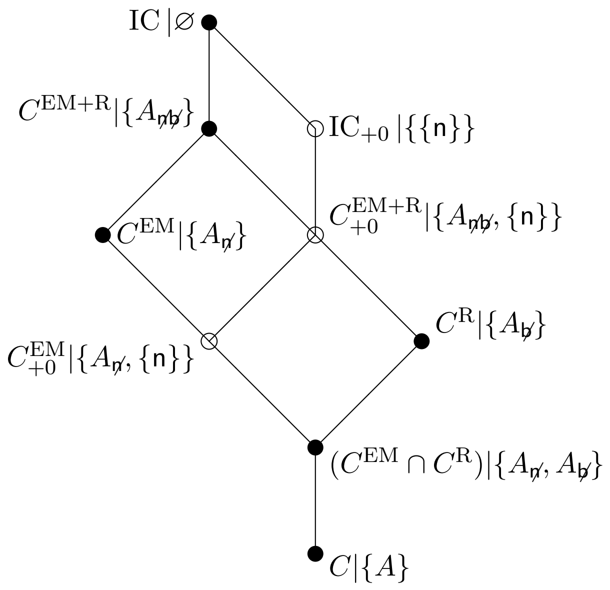

Figure 1. The lattice of $\vee$-disjunctive extensions of $C|$ and their defining matrices.

appears that the $\Sigma^{\supset}_{\sim,+}$-calculus *Pcont* Popov (1989), resulted from *Par* by adding (6.4) and involved therein regardless to any semantics as well, axiomatizes LA. And what is more, it is Theorem 6.19 collectively with Pynko (2000) that have shown that $\mathcal{S}(\downarrow\downarrow/\Downarrow)$/the reservation "being $\veebar$-disjunctive" cannot be, generally speaking, replaced with $\tau_{\veebar}[\mathcal{S}_{\setminus 1}]\downarrow$/omitted in the item (iv/iii) of Theorem 5.16, when taking $C = LP$ and $(\mathcal{S}/\mathcal{C}) = \{(\sim x_0, x_0 \vdash)/(\{\sim x_1, x_1\}|x_0)\}$, respectively. After all, recall that, in view of Theorem 4.1 of Pynko (1995a), $\vee$-disjunctive extensions of B_4 are exactly *De Morgan logics* in the sense of the reference [Pyn 95a] of Pynko (1995b). In this way, the present subsubsection incorporates the material announced therein advancing it much towards arbitrary four-valued expansions, as we briefly outline below.

For instance, when $\Sigma \supseteq \Sigma_{\simeq,+}$ (cf. Subsubsection 6.2.1), in which case $\mathcal{A}$ is $(\neg x_0 \vee x_1)$-implicative, while $\mathbf{S}_*(\mathcal{A}) = \{A[, A_{\not{\mathsf{n}}\not{\mathsf{b}}}]\}$, and so, by Theorem 6.19 and Corollary 5.19, we have:

Corollary 6.20. *Suppose $A_{\not{\mathsf{n}}\not{\mathsf{b}}}$ does not [resp., does] form a subalgebra of $\mathfrak{A}$ [in particular, $\Sigma = \Sigma_{\simeq,+\{,01\}}$]. Then, axiomatic extensions of C are exactly $\vee$-disjunctive ones and form the $(2[+1])$-element chain $C \subsetneq C^{(\mathrm{EM}+)\mathrm{R}} = C^{\mathrm{EM}} = [\mathrm{Cn}^{\omega}_{\mathcal{A}\upharpoonright A_{\not{\mathsf{n}}\not{\mathsf{b}}}} \subsetneq]\,\mathrm{IC}$.*

Likewise, when $\Sigma \supseteq \Sigma_{\sim,2:+}$ (cf. Subsubsection 6.2.2), in which case $\mathbf{S}_*(\mathcal{A}) = \{A[, \{\mathsf{n}\}]\}$, and so, by Theorem 6.19, we get:

Corollary 6.21. *Suppose C does [not] have a theorem. Then, $\vee$-disjunctive extensions of C form the $(2[+1])$-element chain $C[\subsetneq \mathrm{IC}_{+0}] \subsetneq \mathrm{IC} = C^{\mathrm{EM(+R)}}$ with $C^{\mathrm{R}} = \mathrm{IC}_{[+0]}$.*

Finally, when $\Sigma = \Sigma^{\supset}_{\sim,+[,01]}$ (cf. Subsubsection 6.2.3), we set $(\mathcal{DM}/B)^{\supset}_{4[,01]} \triangleq (\mathcal{A}/C)$, in which case we have $\mathbf{S}_*(\mathcal{DM}^{\supset}_{4[,01]}) = \mathbf{S}_*(\mathcal{DM}_{4,01})$, and so, by Theorem 6.19 and Corollary 5.19, we get:

Corollary 6.22. *Suppose $\Sigma \supseteq \Sigma^{\supset}_{\sim,+}$. Then, axiomatic extensions of C are exactly*

$\vee$*-disjunctive ones and form a Galois retract of the six-element non-chain distributive lattice of those of* $B^{\supset}_{4[,01]}$*, depicted at Figure 1 with solely solid circles, and are defined by all those consistent submatrices of* $\mathcal{A}$*, the carriers of whose underlying algebras are subsets of any element of those sets, which mark corresponding nodes, in which case different nodes may correspond to a same extension with different relative axiomatizations, proper consistent ones but* C^{EM} *being axiomatized relatively to* C *by the following axiomatic* Σ*-calculi:*

$$\begin{array}{rcl} (C^{\mathrm{EM}} \cap C^{\mathrm{R}}) & : & \sim x_0 \supset (x_0 \supset (x_1 \vee \sim x_1)), \\ C^{\mathrm{R}} & : & \sim x_1 \supset (x_1 \supset x_0), \\ C^{\mathrm{EM+R}} & : & \{(6.4), (6.7)\}. \end{array} \tag{6.7}$$

6.3. Łukasiewicz finitely-valued logics

Given any $(N \cup \{n\}) \subseteq \omega$, set $(N \div n) \triangleq \{\frac{i}{n} \mid i \in N\}$.

Let $\Sigma \triangleq \{\supset, \neg\}$, $n \in (\omega \setminus 2)$ and $\mathcal{L}_n$ the Σ-matrix with $L_n \triangleq (n \div (n-1))$, $D^{\mathcal{L}_n} \triangleq \{1\}$, $\neg^{\mathcal{L}_n} a \triangleq (1-a)$ and $(a \supset^{\mathcal{L}_n} b) \triangleq \min(1, 1-a+b)$, for all $a, b \in L_n$. The logic $\mathrm{Ł}_n$ of $\mathcal{L}_n$ is known as *Łukasiewicz n-valued logic* (cf. Łukasiewicz (1920) for the three-valued case alone though). By induction on any $m \in (\omega \setminus 1)$, define the secondary unary connective $m\otimes$ of Σ as follows:

$$(m \otimes x_0) \triangleq \begin{cases} x_0 & \text{if } m = 1, \\ \neg x_0 \supset ((m-1) \otimes x_0) & \text{otherwise,} \end{cases}$$

in which case $(m \otimes^{\mathcal{L}_n} a) = \min(1, m \cdot a)$, for all $a \in L_n$, and so, in particular, $(m\otimes)^{\mathcal{L}_n}$ is monotonic. Then, set $(\Box x_0) \triangleq (\neg^{\min(1,n-2)}(n-1) \otimes \neg^{\min(1,n-2)} x_0)$ and $(x_0 \rhd x_1) \triangleq (\Box x_0 \supset \Box x_1)$, being secondary, unless $n = 2$, when $(\Box x_0) = x_0$, and so $\rhd = \supset$ is primary. In that case, $\Box^{\mathcal{L}_n} = ((((n-1) \div (n-1)) \times \{0\}) \cup \{\langle 1, 1 \rangle\})$, and so $\mathcal{L}_n$ is $\rhd$-implicative, for $(\mathcal{L}_n \upharpoonright 2) = \mathcal{L}_2$ is $\supset$-implicative.

And what is more, according to the constructive proof of Proposition 6.10 of Pynko (2009), for each $i \in ((n-1) \setminus 2)$, there is some $\upsilon_i \in \mathrm{Fm}^1_{\{\neg, 2\otimes\}}$ such that $(\upsilon_i^{\mathcal{L}_n}(\frac{i}{n-1}) = 1) \Leftrightarrow (\upsilon_i^{\mathcal{L}_n}(\frac{i-1}{n-1}) \neq 1)$. In addition, put $\upsilon_{n-1} \triangleq x_0 \in \mathrm{Fm}^1_{\{\neg, 2\otimes\}}$ and, in case $n \neq 2$, $\upsilon_1 \triangleq \neg x_0 \in \mathrm{Fm}^1_{\{\neg, 2\otimes\}}$. In this way, for each $i \in (n \setminus 1)$, it holds that $(\upsilon_i^{\mathcal{L}_n}(\frac{i}{n-1}) = 1) \Leftrightarrow (\upsilon_i^{\mathcal{L}_n}(\frac{i-1}{n-1}) \neq 1)$. On the other hand, for every $\upsilon \in \mathrm{Fm}^1_{\{\neg, 2\otimes\}}$, $\upsilon^{\mathcal{L}_n}$ is either monotonic or anti-monotonic, for both $x_0^{\mathcal{L}_n} = \Delta_n$ and $(2\otimes)^{\mathcal{L}_n}$ are monotonic, while $\neg^{\mathcal{L}_n}$ is anti-monotonic. Therefore, for each $i \in N_{0/1} \triangleq \{j \in (n \setminus 1) \mid \upsilon_j^{\mathcal{L}_n}(\frac{j}{n-1}) = / \neq 1\}$, $\upsilon_i^{\mathcal{L}_n}$ is monotonic/anti-monotonic, in which case $(\upsilon_j^{\mathcal{L}_n})^{-1}[\{1\}] = (((n \setminus i) \div (n-1))/(i \div (n-1)))$, respectively, and so $\Upsilon \triangleq \{\upsilon_i \mid i \in (n \setminus 1)\} \supseteq (\{x_0\} \cup \{\neg x_0 \mid n \neq 2\})$ is a finite equality determinant for $\mathcal{L}_n$, $\bar{\upsilon} : (n \setminus 1) \to \Upsilon$ being a bijection supposed to induce a total ordering $\lesssim$ on Υ, in which case $\langle x_0, \neg \rangle = \langle \nu_{n-1}, \neg \rangle$ is not Υ-complex, unless $n = 2$, when all $\langle \Upsilon, \Sigma \rangle$-types are Υ-complex, for, in that case, $\Upsilon = \{x_0\}$. And what is more, as it follows from the constructive proof of Proposition 6.10 of Pynko (2009), non-Υ-complex $\langle \Upsilon, \Sigma \rangle$-types other than $\langle x_0, \neg \rangle$ are exactly those of the form $\langle \upsilon_i, \neg \rangle$, where $\frac{n-1}{2} \geqslant i \in (n \setminus 2)$, and so a $\langle \Upsilon, \Sigma \rangle$-type of the form $\langle \upsilon_i, \neg \rangle$, where $i \in (n \setminus 1)$, is Υ-complex iff $i \in N_c \triangleq \{j \in ((n - \min(1, n-2)) \setminus 1) \mid (j \neq 1) \Rightarrow ((n-1) \in (2 \cdot j))\}$. In particular, in case $n \in (5 \setminus 3)$, $\langle x_0, \neg \rangle$ is the only non-Υ-complex $\langle \Upsilon, \Sigma \rangle$-type. As

$(N_0 \cap N_1) = \varnothing$ and $(N_0 \cup N_1) = (n \setminus 1)$, we have the mapping $\mu \triangleq \{\langle i, k\rangle \in ((n \setminus 1) \times 2) \mid i \in N_k\} : (n \setminus 1) \to 2$.

Let $\mathcal{A} \triangleq \mathcal{L}_n$. Then, $\mathcal{A}'' = \varnothing$. Moreover, under the conventions adopted in both Pynko (2014) and Pynko (2015), we see that both

$$\begin{array}{rcl} \{I_{i-1} : \varphi\} & \leftrightarrow & (\mu(i) : \upsilon_i(\varphi)), \\ \{F_i : \varphi\} & \leftrightarrow & ((1 - \mu(i)) : \upsilon_i(\varphi)), \end{array}$$

where $i \in (n \setminus 1)$ and $\varphi \in \mathrm{Fm}^{\omega}_{\Sigma}$, are true in $\mathcal{A}$. Hence, in view of Corollary 2.4 of Pynko (2014), $\mathcal{A}'''_1 = \{((1-\mu(i)) : \upsilon_i) \uplus (\mu(j) : \upsilon_j) \mid i, j \in (n \setminus 1), i \in j\}$. And what is more, in view of Lemma 2.1 of Pynko (2015), we have a Σ-sequential Υ-table $\mathcal{T}$ for $\mathcal{A}$ given as follows. First, for all $i \in (n \setminus 1)$ and all $m \in 2$, let $\pi_m(\mathcal{T})(\upsilon_i, \neg) \triangleq \{(1-)^{\mu(i)}(1-)^m(1 - \mu(n-i)) : \upsilon_{n-i}\}$. Next, for all $i \in (n \setminus 1)$, let $\pi_{1-\mu(i)}(\mathcal{T})(\upsilon_i, \supset) \triangleq \{(\mu(n-1-k) : \nu_{n-1-k}) \uplus ((1-\mu(i-k)) : \nu_{i-k}(x_1)) \mid k \in i\}$ and $\pi_{\mu(i)}(\mathcal{T})(\upsilon_i, \supset) \triangleq (\{((1-\mu(n-k)) : \upsilon_{n-k}) \uplus (\mu(i-k) : \upsilon_{i-k}(x_1)) \mid k \in (i \setminus 1)\} \cup \{(1-\mu(n-i)) : \upsilon_{n-i}; \mu(i) : \upsilon_i(x_1)\})$. In this way, taking Corollary 3.10(ii) into account, we eventually get:

Corollary 6.23. *Ł$_n$ is axiomatized by the finite calculus $\mathfrak{L}_n$ resulted from $\mathfrak{I}^{\mathrm{PL}}_{\rhd}$ by adding the following axioms:*

$$\upsilon_i \rhd \upsilon_j \qquad (\langle i, j\rangle \in ((\ker \mu) \cap (\in \cap\, n^2)^{(2\cdot\mu(i))-1})$$

$$\upsilon_i \mathrel{\underline{\vee}_{\rhd}} \upsilon_j \qquad (\langle i, j\rangle \in (\mu^{-1}[\in \cap\, 2^2] \cap (\in \cap\, n^2))$$

$$\upsilon_i \rhd (\upsilon_j \rhd x_1) \qquad (\langle i, j\rangle \in (\mu^{-1}[\ni \cap\, 2^2] \cap (\in \cap\, n^2))$$

$$\upsilon_{n-i} \mathrel{\underline{\vee}_{\rhd}} \upsilon_i(\neg x_0) \qquad (i \in N_c, \mu(i) = \mu(n-i))$$

$$\upsilon_{n-i} \rhd (\upsilon_i(\neg x_0) \rhd x_1) \qquad (i \in N_c, \mu(i) = \mu(n-i))$$

$$\upsilon_{n-i} \rhd \upsilon_i(\neg x_0) \qquad (i \in N_c, \mu(i) \neq \mu(n-i))$$

$$\upsilon_i(\neg x_0) \rhd \upsilon_{n-i} \qquad (i \in N_c, \mu(i) \neq \mu(n-i))$$

$$\upsilon_{n-1-k} \rhd (\upsilon_{i-k}(x_1) \rhd (\upsilon_i(x_0 \supset x_1) \rhd x_2)) \qquad (k \in i \in (n \setminus 1), \mu(i) = \mu(n-1-k) = 0 \neq \mu(i-k))$$

$$\upsilon_{n-1-k} \rhd (\upsilon_i(x_0 \supset x_1) \rhd \upsilon_{i-k}(x_1)) \qquad (n \neq 2, k \in i \in (n \setminus 1), \mu(i) = \mu(n-1-k) = 0 = \mu(i-k))$$

$$\upsilon_{n-1-k} \rhd (\upsilon_{i-k}(x_1) \rhd \upsilon_i(x_0 \supset x_1)) \qquad (k \in i \in (n \setminus 1), \mu(i) \neq \mu(n-1-k) = 0 \neq \mu(i-k))$$

$$\upsilon_{i-k}(x_1) \rhd (\upsilon_i(x_0 \supset x_1) \rhd \upsilon_{n-1-k}) \qquad (k \in i \in (n \setminus 1), \mu(i) = 0 \neq \mu(n-1-k) = \mu(i-k))$$

$$(\upsilon_{n-1-k} \mathrel{\underline{\vee}_{\rhd}} \upsilon_{i-k}(x_1)) \mathrel{\underline{\vee}_{\rhd}} \upsilon_i(x_0 \supset x_1) \qquad (k \in i \in (n \setminus 1), \mu(i) = \mu(n-1-k) = 1 \neq \mu(i-k))$$

$$(\upsilon_{n-1-k} \rhd x_2) \rhd ((\upsilon_{i-k}(x_1) \rhd x_2) \rhd (\upsilon_i(x_0 \supset x_1) \rhd x_2)) \qquad (k \in i \in (n \setminus 1), \mu(i) = 0 = \mu(i-k) \neq \mu(n-1-k))$$

$$(\upsilon_{n-1-k} \rhd x_2) \rhd ((\upsilon_i(x_0 \supset x_1) \rhd x_2) \rhd (\upsilon_{i-k}(x_1) \rhd x_2)) \qquad (k \in i \in (n \setminus 1), \mu(i) = 1 = \mu(n-1-k) = \mu(i-k))$$

$$(\upsilon_{i-k}(x_1) \rhd x_2) \rhd ((\upsilon_i(x_0 \supset x_1) \rhd x_2) \rhd$$

$$(\upsilon_{n-1-k} \rhd x_2)) \qquad (k \in i \in (n \setminus 1), \mu(i) \neq 0 = \mu(n-1-k) = \mu(i-k))$$

$$\upsilon_{n-k} \rhd (\upsilon_{i-k}(x_1) \rhd (\upsilon_i(x_0 \supset x_1) \rhd x_2)) \qquad (i \in (n \setminus 1), k \in (i \setminus 1), \mu(i) = \mu(n-k) = 1 \neq \mu(i-k))$$

$$\upsilon_{n-k} \rhd (\upsilon_{i-k}(x_1) \rhd \upsilon_i(x_0 \supset x_1)) \qquad (i \in (n \setminus 1), k \in (i \setminus 1), \mu(i) \neq \mu(n-k) = 1 \neq \mu(i-k))$$

$$\upsilon_{n-k} \rhd (\upsilon_i(x_0 \supset x_1) \rhd \upsilon_{i-k}(x_1)) \qquad (i \in (n \setminus 1), k \in (i \setminus 1), \mu(i) = \mu(n-k) = 1 = \mu(i-k))$$

$$\upsilon_{i-k}(x_1) \rhd (\upsilon_i(x_0 \supset x_1) \rhd \upsilon_{n-k}) \qquad (i \in (n \setminus 1), k \in (i \setminus 1), \mu(i) \neq \mu(n-k) = 0 = \mu(i-k))$$

$$(\upsilon_{n-k} \veebar_{\rhd} \upsilon_{i-k}(x_1)) \veebar_{\rhd} \upsilon_i(x_0 \supset x_1) \qquad (i \in (n \setminus 1), k \in (i \setminus 1), \mu(i) = \mu(n-k) = 0 \neq \mu(i-k))$$

$$(\upsilon_{n-k} \rhd x_2) \rhd ((\upsilon_{i-k}(x_1) \rhd x_2) \rhd (\upsilon_i(x_0 \supset x_1) \rhd x_2)) \qquad (i \in (n \setminus 1), k \in (i \setminus 1), \mu(i) \neq \mu(n-k) = 0 \neq \mu(i-k))$$

$$(\upsilon_{n-k} \rhd x_2) \rhd ((\upsilon_i(x_0 \supset x_1) \rhd x_2) \rhd (\upsilon_{i-k}(x_1) \rhd x_2)) \qquad (i \in (n \setminus 1), k \in (i \setminus 1), \mu(i) = \mu(n-k) = 0 = \mu(i-k))$$

$$(\upsilon_{i-k}(x_1) \rhd x_2) \rhd ((\upsilon_i(x_0 \supset x_1) \rhd x_2) \rhd (\upsilon_{n-k} \rhd x_2)) \qquad (i \in (n \setminus 1), k \in (i \setminus 1), \mu(i) \neq \mu(n-k) = 1 = \mu(i-k))$$

$$\upsilon_{n-i} \rhd \upsilon_i(x_0 \supset x_1) \qquad (i \in N_0 \not\ni (n-i))$$

$$\upsilon_i(x_0 \supset x_1) \rhd \upsilon_{n-i} \qquad (i \in N_1 \not\ni (n-i))$$

$$\upsilon_{n-i} \rhd (\upsilon_i(x_0 \supset x_1) \rhd x_2) \qquad (i \in N_1 \ni (n-i))$$

$$\upsilon_{n-i} \veebar_{\rhd} \upsilon_i(x_0 \supset x_1) \qquad (n \neq 2, i \in N_0 \ni (n-i))$$

$$\upsilon_i(x_1) \rhd \upsilon_i(x_0 \supset x_1) \qquad (n \neq 2, i \in N_0)$$

$$\upsilon_i(x_0 \supset x_1) \rhd \upsilon_i(x_1) \qquad (i \in N_1)$$

It is remarkable that, in the classical case, when $n = 2$, the additional axioms of $\mathfrak{L}_n$ are exactly the Excluded Middle Law axiom $(x_0 \veebar_{\rhd} \neg x_0) = ((x_0 \supset \neg x_0) \supset \neg x_0)$ and the Ex Contradictione Quodlibet axiom $x_0 \supset (\neg x_0 \supset x_1)$, $\mathfrak{L}_2$ being a well-known natural Hilbert-style axiomatization of the classical logic. And what is more, $\mathfrak{L}_n$ grows just polynomially (more precisely, quadratically) on n, so it eventually looks relatively good, the additional axioms of $\mathfrak{L}_3$ being as follows, where $i \in 2$:

$$\begin{array}{lcr}
\neg x_1 \rhd (x_1 \rhd x_0) & \neg^i x_i \rhd ((x_0 \supset x_1) \rhd \neg^i x_{1-i}) & \neg x_0 \rhd (x_0 \supset x_1) \\
x_0 \rhd \neg\neg x_0 & x_0 \rhd (\neg x_1 \rhd \neg(x_0 \supset x_1)) & x_1 \rhd (x_0 \supset x_1) \\
\neg\neg x_0 \rhd x_0 & (x_0 \veebar_{\rhd} \neg x_1) \veebar_{\rhd} (x_0 \supset x_1) & \neg(\neg x_0 \supset x_1) \rhd \neg x_1
\end{array}$$

Concluding this discussion, we should like to highlight that, though, in general, an analytical expression (if any, at all) for $\bar{\upsilon}$ has not been known yet, the constructive proof of Proposition 6.10 of Pynko (2009) has been implemented upon the basis of SCWI-Prolog resulting in a quite effective logical program (taking less than second up to $n = 1000$) calculating $\bar{\upsilon}$, and so immediately yielding definitive explicit formulations of both $\mathcal{T}$ (in particular, of the Gentzen-style axiomatization $\mathcal{S}_{\mathcal{A},\mathcal{T}}^{(0,0)}$ of Ł_n; cf. Pynko (2004)) and the Hilbert-style axiomatization $\mathfrak{L}_n$ of Ł_n found above.

6.4. Hałkowska-Zajac logic

Here, it is supposed that $\Sigma \triangleq \Sigma_{\sim,+}$, $(\mathfrak{A}{\restriction}\Sigma_+) \triangleq \mathfrak{D}_3$, $\sim^{\mathfrak{A}} i \triangleq (\min(1,i)\cdot(3-i))$, for all $i \in 3$, and $D^{\mathcal{A}} \triangleq \{0,2\}$, in which case $\mathcal{A}$, defining the logic HZ Hałkowska and Zajac (1988), is $\supset$-implicative, where $(x_0 \supset x_1) \triangleq ((\sim x_0 \wedge \sim x_1) \vee x_1)$ is secondary, while $\{x_0, \sim x_0\}$ is an equality determinant for $\mathcal{A}$ (cf. Example 2 of Pynko (2004)), and so $\mathcal{A}'' = \varnothing$ and $\mathcal{A}''' = \{\vdash \sim x_0, x_0\}$. First, we have $\sim^{\mathfrak{A}}\sim^{\mathfrak{A}} a = a$, for all $a \in A$. Therefore, one can take $\lambda_{\mathcal{T}}(\sim x_0, \sim) = \{x_0 \vdash\}$ and $\rho_{\mathcal{T}}(\sim x_0, \sim) = \{\vdash x_0\}$ to satisfy (5.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(\sim x_0, \sim) = \{\sim\sim x_0 \vdash x_0\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(\sim x_0, \sim) = \{x_0 \vdash \sim\sim x_0\}$. Next, consider any $a, b \in A$. Then, $\sim^{\mathfrak{A}}(a(\wedge/\vee)^{\mathfrak{A}} b) \in D^{\mathcal{A}}$ iff either/both $\sim^{\mathfrak{A}} a \in D^{\mathcal{A}}$ or/and $\sim^{\mathfrak{A}} b \in D^{\mathcal{A}}$. Therefore, one can take $\lambda_{\mathcal{T}}(\sim x_0, \vee) = \{\sim x_0, \sim x_1 \vdash\}$ and $\rho_{\mathcal{T}}(\sim x_0, \vee) = \{\vdash \sim x_0; \vdash \sim x_1\}$ to satisfy (5.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(\sim x_0, \vee) = \{\sim(x_0 \vee x_1) \vdash \sim x_0; \sim(x_0 \vee x_1) \vdash \sim x_1\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(\sim x_0, \vee) = \{\sim x_0, \sim x_1 \vdash \sim(x_0 \vee x_1)\}$. Likewise, one can take $\lambda_{\mathcal{T}}(\sim x_0, \wedge) = \{\sim x_0 \vdash; \sim x_1 \vdash\}$ and $\rho_{\mathcal{T}}(\sim x_0, \wedge) = \{\vdash \sim x_0, \sim x_1\}$ to satisfy (5.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(\sim x_0, \wedge) = \{\sim(x_0 \wedge x_1) \vdash \sim x_0, \sim x_1\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(\sim x_0, \wedge) = \{\sim x_0 \vdash \sim(x_0 \wedge x_1); \sim x_1 \vdash \sim(x_0 \wedge x_1)\}$. Moreover, $(a(\wedge/\vee)^{\mathfrak{A}} b) \in D^{\mathcal{A}}$ iff both $(a = 1) \Rightarrow (b = (0/2))$ and $(b = 1) \Rightarrow (a = (0/2))$. Therefore, one can take $\rho_{\mathcal{T}}(x_0, \wedge) = \{\vdash x_0, x_1; \vdash \sim x_0, x_1; \vdash \sim x_1, x_0\}$ and $\lambda_{\mathcal{T}}(x_0, \wedge) = \{x_0, x_1 \vdash; x_0, \sim x_0 \vdash; x_1, \sim x_1 \vdash\}$ to satisfy (5.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(x_0, \wedge) = \{(x_0 \wedge x_1) \vdash x_0, x_1; (x_0 \wedge x_1) \vdash \sim x_0, x_1; (x_0 \wedge x_1) \vdash \sim x_1, x_0\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(x_0, \wedge) = \{x_0, x_1 \vdash (x_0 \wedge x_1); x_0, \sim x_0 \vdash (x_0 \wedge x_1); x_1, \sim x_1 \vdash (x_0 \wedge x_1)\}$. Likewise, one can take $\rho_{\mathcal{T}}(x_0, \vee) = \{\vdash x_0, x_1; \sim x_1 \vdash x_0; \sim x_0 \vdash x_1\}$ and $\lambda_{\mathcal{T}}(x_0, \vee) = \{x_0, x_1 \vdash; \vdash \sim x_0; \vdash \sim x_1\}$ to satisfy (5.1), in which case $\boldsymbol{\lambda}_{\mathcal{T}}(x_0, \vee) = \{(x_0 \vee x_1) \vdash x_0, x_1; \sim x_1, (x_0 \vee x_1) \vdash x_0; \sim x_0, (x_0 \vee x_1) \vdash x_1\}$ and $\boldsymbol{\rho}_{\mathcal{T}}(x_0, \vee) = \{x_0, x_1 \vdash (x_0 \vee x_1); \vdash \sim x_0, (x_0 \vee x_1); \vdash \sim x_1, (x_0 \vee x_1)\}$. In this way, taking Corollary 3.10(ii) into account, we eventually get:

Corollary 6.24. *HZ is axiomatized by the calculus $\mathcal{HZ}$ resulted from $\mathcal{I}_{\supset}^{\mathrm{PL}}$ by adding the axioms* (6.1), (6.2), (6.3) *and the following ones, where $i \in 2$:*

$$
\begin{array}{lr}
(x_0 \supset x_2) \supset ((x_1 \supset x_2) \supset ((x_0 \wedge x_1) \supset x_2)) & x_0 \supset (x_1 \supset (x_0 \wedge x_1)) \\
(\sim x_i \supset x_2) \supset ((x_{1-i} \supset x_2) \supset ((x_0 \wedge x_1) \supset x_2)) & x_i \supset (\sim x_i \supset (x_0 \wedge x_1)) \\
(x_0 \supset x_2) \supset ((x_1 \supset x_2) \supset ((x_0 \vee x_1) \supset x_2)) & x_0 \supset (x_1 \supset (x_0 \vee x_1)) \\
(\sim x_i \supset (x_0 \vee x_1)) \supset (x_0 \vee x_1) & \sim x_{1-i} \supset ((x_0 \vee x_1) \supset x_i) \\
 & (\sim x_0 \supset x_0) \supset x_0
\end{array}
$$

In this connection, recall that an *infinite* Hilbert-style axiomatization of HZ has been due to Zbrzezny (1990).

7. Conclusions

As a matter of fact, Subsection 6.2 has provided finite Hilbert-style axiomatizations of *all* miscellaneous expansions of B_4 studied in Pynko (1999) and their disjunctive extensions (in this connection, it is remarkable that we have avoided any guessing their relative axiomatizations right — though such would not be difficult, as it has originally been done in the reference [Pyn 95 a] of Pynko (1995b) — but rather have just manually followed analytical expressions we have found to demonstrate their practical applicability to effective/computational finding "good" relative axiomatizations in other more complicated cases like Łukasiewicz logics). Even though Section 6 does not exhaust *all* interesting applications of Section 5, it has definitely incorporated *most acute* ones. In general, the effective nature of the present elaboration definitely makes the paper a part of *Applied* Non-Classical Logic, especially due to quite effective program implementations invented in this connection.

References

Asenjo, F. G., & Tamburino, J. (1975). Logic of antinomies. *Notre Dame Journal of Formal Logic*, *16*, 272–278.

Belnap, N. D., Jr. (1977). A useful four-valued logic. In J. M. Dunn & G. Epstein (Eds.), *Modern uses of multiple-valued logic* (pp. 8–37). Dordrecht: D. Reidel Publishing Company.

Dyrda, K., & Prucnal, T. (1980). On finitely based consequence determined by a distributive lattice. *Bulletin of the Section of Logic*, *9*, 60–66.

Hałkowska, K., & Zajac, A. (1988). O pewnym, trójwartościowym systemie rachunku zdań. *Acta Universitatis Wratislaviensis. Prace Filozoficzne*, *57*, 41–49.

Kleene, S. C. (1952). *Introduction to metamathematics.* New York: D. Van Nostrand Company.

Loś, J., & Suszko, R. (1958). Remarks on sentential logics. *Indagationes Mathematicae*, *20*, 177–183.

Łukasiewicz, J. (1920). O logice trójwartościowej. *Ruch Filozoficzny*, *5*, 170–171.

Mal'cev, A. I. (1965). *Algebraic systems.* New York: Springer Verlag.

Mendelson, E. (1979). *Introduction to mathematical logic* (2nd ed.). New York: D. Van Nostrand Company.

Peirce, C. (1885). On the Algebra of Logic: A Contribution to the Philosophy of Notation. *American Journal of Mathematics*, *7*, 180-202.

Popov, V. M. (1989). Sequential formulations of some paraconsistent logical systems. In V. A. Smirnov (Ed.), *Syntactic and semantic investigations of non-extensional logics* (pp. 285–289). Moscow: Nauka. (In Russian)

Priest, G. (1979). The logic of paradox. *Journal of Philosophical Logic*, *8*, 219–241.

Pynko, A. P. (1995a). Characterizing Belnap's logic via De Morgan's laws. *Mathematical Logic Quarterly*, *41*(4), 442–454.

Pynko, A. P. (1995b). On Priest's logic of paradox. *Journal of Applied Non-Classical Logics*, *5*(2), 219–225.

Pynko, A. P. (1999). Functional completeness and axiomatizability within Belnap's four-valued logic and its expansions. *Journal of Applied Non-Classical Logics*, *9*(1/2), 61–105. (Special Issue on Multi-Valued Logics)

Pynko, A. P. (2000). Subprevarieties versus extensions. Application to the logic of paradox. *Journal of Symbolic Logic*, *65*(2), 756–766.

Pynko, A. P. (2004). Sequential calculi for many-valued logics with equality determinant. *Bulletin of the Section of Logic*, *33*(1), 23–32.

Pynko, A. P. (2009). Distributive-lattice semantics of sequent calculi with structural rules. *Logica Universalis*, *3*(1), 59–94.

Pynko, A. P. (2014). Minimal sequent calculi for monotonic chain finitely-valued logics. *Bulletin of the Section of Logic*, *43*(1/2), 99–112.

Pynko, A. P. (2015). Minimal Sequent Calculi for Łukasiewicz's Finitely-Valued Logics. *Bulletin of the Section of Logic*, *44*(3/4), 149–153.

Zbrzezny, A. (1990). The Hilbert-type axiomatization of some three-valued propositional logic. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, *36*, 415–421.