On Hadamard Matrices

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ABSTRACT

In this research paper, the relationship between symmetric unit hypercube and Hadamard matrices is briefly discussed. Also, synthesis of Hadamard matrices of higher order from non-Hadamard matrices of lower order is explored. Some novel ideas on construction of Hadamard martrices are proposed.

1. INTRODUCTION:

Linear Algebra, as a branch of mathematics started in an effort to solve linear system of equations with coefficients belonging to the field of real/complex numbers. Matrices/Linear operators provided concrete formulation for specifying such a problem. Mathematicians, such as Toeplitz, Laplace, Hankel naturally identified "structured" matrices which arise in specific research problems in linear algebra. Motivated by such pioneering efforts, Sylvester and Hadamard independently identified an interesting class of structured matrices. It was realized by later generations of researchers that Hadamard matrices find various applications in research areas such as the theory of error correcting codes.

The author discovered that Hadamard matrices naturally arise in the synthesis of Hopfield Associative Memory (HAM), a type of Artificial Neural Network (ANN). This relationship between Hadamard matrices and Hopfield Neural Network (HNN) motivated the author to study novel methods of construction of Hadamard matrices. Such a humble research effort resulted in this research paper.

This research paper is organized as follows. In Section 2, known research literature is reviewed. In Section 3, the relationship between symmetric unit hypercube and Hadamard matrices is discussed. In Section 4, some novel constructions of Hadamard matrices are explored. The research paper concludes in Section 5.

2. REVIEW OF RESEARCH LITERATURE:

We start the review with the definition of Hadamard matrix:

Definition: A Hadamard marix of order 'm', denoted by H_m, is an m x m matrix of +1's and -1's such that

$$H_m H_m^T = m I_m, \text{ where}$$

I_m is an m x m identity matrix. The above definition is equivalent to saying that any two rows of H_m are orthogonal. Hadamard matrices of order 2^k exist for all $k \geq 0$. The so-called Sylvester construction is as follows:

$$H_1 = 1.$$

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

$$H_{2^{m+1}} = \begin{bmatrix} H_{2^m} & H_{2^m} \\ H_{2^m} & -H_{2^m} \end{bmatrix} \text{ for integer } m \geq 2.$$

Note: *For the sake of simplicity, we consider symmetric Hadamard matrices in the following discussion (as and when they are required) i.e. We look for { +1, 1 } matrices whose columns are orthogonal (instead of rows).*

In combinatorial terms, the definition of Hadamard matrix means that each pair of rows has matching entries in exactly half of their columns and mismatched entries in the remaining columns. It is a consequence of this definition that the corresponding properties hold for columns as well as rows.

Hadamard made several interesting contributions to the theory of Hadamard matrices:

- **Hadamard's Maximal Determinant Problem**: Hadamard matrix has maximal determinant among matrices with entries of absolute value less than or equal to 1.

- **Hadamard Conjecture**: Hadamard matrix of order 4k exists for every positive integer k.

- Hadamard matrices of order 12 and 20 were constructed by Hadamard (in 1893).

A generalization of Sylvester construction proves that if H_n and H_m are Hadamard matrices of orders 'n' and 'm' respectively, then Kronecker product of H_n and H_m, is a Hadamard matrix of order nm. This result is used to produce Hadamard matrices of higher order once those of smaller order are known. Payley discovered a construction which enables construction of certain Hadamard matrices.

In 2005, Hadi Kharaghani and Behruz Tayfeh-Rezaie published their construction of Hadamard Matrix of order 428. As a result, the smallest order for which no Hadamard matrix is presently known is 668.

3. SYMMETRIC UNIT HYPERCUBE: HADAMARD MATRICES:

Symmetric unit hypercube naturally arises in many interesting applications. It can be defined in the following manner.

Definition: Let $\bar{X}$ be an N x 1 vector (i.e. a column vector). $\bar{L}$ (Symmetric Unit Hypercube) is a subset of N-dimensional Euclidean space, defined as $\overline{L_N} = \{ \bar{X} = (x_1 \ x_2 \ldots . x_N)^T : \ \ x_i = +1 \ or - 1 \}$ i.e. all N-dimensional vectors whose components can only assume the values '+1' or '-1'.

From the above definition, it is clear that the 2^N points on the symmetric unit hypercube are symmetrically located about the origin.

Now the following Lemma is very helpful in the study of Hadamard matrices.

Lemma 1: Let X, Y be vectors on the set H. Then the inner product of X, Y i.e. $< X, Y >$ = N – 2 $d_H\,(X, Y\,)$, where $d_H\,(X, Y\,)$ is the integer valued Hamming Distance between the vectors $X, Y \in H$.

Proof: Refer Lemma 2 in [RaM].

From the above discussion, it is clear that the rows/columns of Hadamard matrix are those vectors on H that are orthogonal to each other i.e. $< X, Y >$ = 0. Thus, from the above lemma, we have that

$$< X, Y > = 0 \text{ if and only if } d_H\,(X, Y\,) = \frac{N}{2} .$$

Thus, for $d_H\,(X, Y\,)$ to be an integer, N must be an even number. Hence, we have the following well known conclusion.

Claim: Hadamard matrices exist only in an even dimension that are multiples of 4.

Note: We consider the case where N is an even number. From Lemma 1, it readily follows that if $d_H\,(X, Y\,) \in \left\{ 0,1,2, \ldots , \frac{N}{2} \right\}$, *then*

$$< \ X, Y \ > \in \{ \mathrm{N}, \mathrm{N}-2, \ldots , 0 \} \ and \ \ if \ \ d_H\,(X, Y\,) \ \in \left\{ \frac{N}{2} + 1, \frac{N}{2} + 2, \ldots , N \ \right\}, then$$

$$< \ X, Y \ > \in \{ -2, -4, \ldots , -\mathrm{N} \}$$

Note: The problem of constructing Hadamard matrices in certain even dimension is surprisingly difficult problem. The author applied the construction of Hadamard matrices in synthesis of Hopfield Neural Network [RaM], [Rama2].

Sylvester designed an ingenious method of constructing Hadamard matrices in dimension N, which is such that $N = 2^m, where\ 'm'\ is\ an\ integer$. Such construction was explained in Section 2.

The problem of constructing new Hadamard matrices has been attempted by several researchers such as Payley. The innovative efforts of the author are summarized in the next section.

4. NOVEL CONSTRUCTION OF HADAMARD MATRICES:

As a natural beginning, the author identified Hadamard matrices and non-Hadamard matrices of dimension 2. They are provided below:

- Hadamard Matrices of Dimension 2:

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}.$$

$It\ can\ be\ noticed\ that\ the\ 2\,x\,2\ Hadamard\ matrices\ in\ second\ row\ are\ negative\ of\ the\ corresponding\ 2\,x\,2\ matrices\ in\ the\ first\ row.$

- Non-Hadamard Matrices of Dimension 2:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}.$$

Note: We readily realize that if H_N is a Hadamard matrix of dimension N, then $-H_N$ is also Hadamard matrix.

Definition: For every pair of Hadamard matrices $\{ H_N, -H_N \}$, we call H_N as a primitive Hadamard matrix.

Note: We can similarly define "primitive" non-Hadamard matrices of any dimension N.

- We now introduce an interesting arithmetical function in the spirit of arithmetical functions in number theory:

$$\vartheta(N): \textit{Number of "Primitive" Hadamard Matrices of Dimension } N.$$

$$\textit{In view of Hadamard Conjecture, congruence properties of } \vartheta(N) \textit{ can be studied.}$$

$$\textit{From the above discussion,} \quad \vartheta(2) = 4 \textit{ and } \vartheta(N) = 0 \textit{ if } N \textit{ is an odd number.}$$

Note: An interesting research problem is to determine $\vartheta(N)$ *for an arbitrary* N.

Note: In known literature the following terminology is used.

- Two Hadamard matrices are considered equivalent if one can be obtained from the other by negating rows or columns, or by interchanging rows or columns. Up to equivalence, there is a unique Hadamard matrix of orders 1, 2, 4, 8, and 12.

- Traditionally, Hadamard matrices of Higher dimension are constructed from Hadamard matrices of lower dimension (as in the case of Sylvester construction *). In contrast, we now illustrate the construction of Hadamard matrices of dimension 4 from STRUCURED NON-HADAMARD matrices of dimension 2*.

$$\widehat{H_4} = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \; and \; -\widehat{H_4}\,.$$

We now compare the above 4 dimensional Hadamard matrix with the one constructed from Sylvester construction (in which a Hadamard matrix of 2 dimension is utilized).

$$H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \; and \; -H_4.$$

It can be readily seen that the 4-dimensional symmetric unit hypercube i.e. L_4 is

$$L_4 = H_4 \cup -H_4 \cup \widehat{H_4} \cup -\widehat{H_4}\,.$$

- In view of the construction of $\widehat{H_4}$, we introduce, the following interesting concept of utility in synthesizing higher dimensional Hadamard matrices from certain lower dimensional { +1, -1 } matrices which are not Hadamard matrices.

Definition: Two { +1, -1 } matrices A, B of dimension N are said to be ORTHOGONAL if every pair of column vectors from each of them are orthogonal to each other i.e. Let

$\{ X_i , Y_i \ for \ 1 \leq i \leq N \}$ be the columns of A, B respectively. Then

$< X_i, Y_j >$ = 0 $\ for \ all \ i, j$ (i.e. the inner product of any pair of columns from them is ZERO) i.e. Hamming distance from any pair of columns from A, B is $\frac{N}{2}$.

- It can be easily seen that, there are NO ORTHOGONAL Hadamard matrices of dimension 2. Since columns of Hadamard matrices form basis of R^{4N} (4N dimensional Euclidean space), there are no orthogonal Hadamard matrices (for each N). But there are 4 "primitive" non-Hadamard orthogonal matrices of dimension 2.

The following definition enables construction of Hadamard matrices:

Definition: Two { +1, -1 } matrices A, B of dimension N are said to be STRUCTURED, ORTHOGONAL matrices if the following condition is satisfied i.e. $if \ < A_i, A_j > = \ J,$ $then < B_{i,}, B_j > = \ -J \ (\ including \ J = 0 \ case)$ ($A_i, B_i \ are \ columns \ of \ matrices \ A, B$).

Note: In most structured, orthogonal Non-Hadamard matrices $J = N \ or \ J = \ 0$.

The synthesis of $\widehat{H_4}$, naturally provides the following approach to construct higher dimensional Hadamard matrices from lower dimensional { +1, -1 } matrices which are not necessarily Hadamard matrices. Let $\{ B_N , \ C_N \}$ be "ORTHOGONAL and STRUCTURED" Non-Hadamard matrices of dimension N.

$$H_{2N} = \begin{bmatrix} B_N & C_N \\ C_N & B_N \end{bmatrix} \ which \ is \ a \ Hadamard \ matrix \ of \ dimension \ 2N.$$

We have another 4-dimensional Hadamard matrix, in the spirit of above construction

(i.e. like $\widehat{H_4}$ synthesized from orthogonal non-Hadamard { +1, -1 } 2 x 2 matrices)

$$\widetilde{H_4} = \begin{bmatrix} -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ -1 & 1 & -1 & -1 \\ -1 & 1 & 1 & 1 \end{bmatrix}.$$

Note: Upto sign change, there are only two 4 dimensional Hadamard matrices ($\widehat{H_4}, \widetilde{H_4}$) constructed from two orthogonal non-Hadamard matrices of dimension 2 i.e. not every pair of orthogonal non-Hadamard matrices of dimension 2 lead to Hadamard matrix of dimension 4.

- We now provide example of 8-dimensional Hadamard matrix, using $\widehat{H_4}$ and $\widetilde{H_4}$, in the spirit of above construction

$$H_8 = \begin{bmatrix} \widehat{H}_4 & \widetilde{H}_4 \\ \widetilde{H}_4 & \widehat{H}_4 \end{bmatrix}.$$

The above example naturally leads to the following question

Q: Are there 4 x 4 NON-HADAMARD, STRUCTURED { +1, -1 } matrices, that lead to 8 x 8 matrix using the above construction?

Answer: Yes. Consider the following 4 x 4 , ORTHOGONAL NON-HADAMARD matrices.

$$B = \begin{bmatrix} -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & 1 & 1 & -1 \end{bmatrix} \text{ and } C = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

It can be easily seen that $\begin{bmatrix} B & C \\ C & B \end{bmatrix}$ is an 8 x 8 Hadamard matrix. Also, the following "orthogonal" Non-Hadamard matrices lead to an 8 x 8 Hadamard matrix (as above)

$$B = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}.$$

Note: Let $\{ X_i , \ Y_i \ for \ 1 \leq i \leq N \}$ be the columns of B, C respectively. The orthogonal non-Hadamard matrices A, B are STRUCTURED/CONSTRAINED in the sense that if

$< X_i, X_j >$ = S, then $< Y_i, Y_j > = -S$ (i.e. the inner product of their columns/rows are opposite in value including ZERO).

- The above construction naturally suggests the following Lemma

Lemma 2 : Let $\{ B_N , \ C_N \}$ be STRUCTURED "ORTHOGONAL" Non-Hadamard matrices of dimension N. Then, the following block symmetric matrix of dimension '2N' is a Hadamard matrix (which need not be symmetric).

$$H_{2N} = \begin{bmatrix} B_N & C_N \\ C_N & B_N \end{bmatrix}.$$

Proof: Let $B_N = [\, b_1 \quad b_2 \quad \dots \quad b_{N-1} \quad b_N \,]$ i.e. $b_i^{\prime s}$ are columns of Structured Orthogonal Non-Hadamard matrix, B_N.

Similarly, let $C_N = [\, c_1 \quad c_2 \quad \cdots \quad c_{N-1} \quad c_N \,]$. From the definition of STRUCTURED ORTHOGONAL Non-Hadamard matrices, the result readily follows i.e. H_{2N} is a Hadamard matrix of dimension 2N. Q.E.D.

Note: Based on combinatorial arguments, the number of vectors on symmetric, unit hypercube which are at a Hamming distance { 0,1,2,.., N } from each other can easily be determined. Such a result confirms the existence of structured, orthogonal, non-Hadamard matrices in dimension N, such that N (mod) 4 = 0.

Note: In attacking the Hadamard conjecture, it is sufficient to reason that atleast a pair of structured, orthogonal non-Hadamard matrices exist in each dimension N such that N (mod) 4 = 0.

Note: In any dimension which is a multiple of 4, structured, orthogonal non-Hadamard matrices can be readily constructed (and always exist). Hence the above Lemma enables construction of higher dimensional Hadamard matrices from them

5. CONCLUSIONS:

In this research paper, the possibility of synthesizing Hadamard matrices of Higher order from non-Hadamard matrices of lower order is explored. Some novel ideas on construction of Hadamard matrices are explored.

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