Catalan's Constant is Irrational

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November 14, 2022

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Abstract

In mathematics, Catalan's constant G is defined by

$$G = \beta(2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \cdots ,$$

where β is the Dirichlet beta function.

Catalan's constant has been called arguably the most basic constant whose irrationality and transcendence (though strongly suspected) remain unproven. In this paper we show that G is indeed irrational.

Proof

Keeping in mind the Riemann series theorem (also called the Riemann rearrangement theorem), we have

$$\begin{array}{cccccc|l}
\frac{1}{1^2} & -\frac{1}{3^2} & +\frac{1}{5^2} & -\frac{1}{7^2} & +\frac{1}{9^2} & -\cdots & G \\
 & -\frac{2}{3^2} & +\frac{2}{5^2} & -\frac{2}{7^2} & +\frac{2}{9^2} & -\cdots & 2G - \frac{2}{1^2} \\
 & & +\frac{2}{5^2} & -\frac{2}{7^2} & +\frac{2}{9^2} & -\cdots & 2G - \frac{2}{1^2} + \frac{2}{3^2} \\
 & & & -\frac{2}{7^2} & +\frac{2}{9^2} & -\cdots & 2G - \frac{2}{1^2} + \frac{2}{3^2} - \frac{2}{5^2} \\
 & & & & +\frac{2}{9^2} & -\cdots & 2G - \frac{2}{1^2} + \frac{2}{3^2} - \frac{2}{5^2} + \frac{2}{7^2} \\
 & & & & & \cdots & \cdots \\
\hline
\frac{1}{1} & -\frac{1}{3} & +\frac{1}{5} & -\frac{1}{7} & +\frac{1}{9} & -\ldots &
\end{array}$$

Notice that the Leibniz formula for π states that

$$\frac{\pi}{4} = \beta(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots .$$

Moreover, it is easy to see that $\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$ is conditionally convergent. On the another hand, $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$ is **absolutely convergent and we are able to rearrange the terms as we want.**

Let's assume **the contrary**: G is a rational number $\frac{s}{2^k t}$, where t is **odd**. Hence, we have

$$stG = st \sum_{n=0,\ (2n+1)\nmid t}^{\infty} \frac{(-1)^n}{(2n+1)^2} + st \sum_{m=0}^{\infty} \frac{(-1)^{mt+\lfloor t/2 \rfloor}}{t^2(2m+1)^2} =$$

$$st \sum_{n=0,\ (2n+1)\nmid t}^{\infty} \frac{(-1)^n}{(2n+1)^2} + ((-1)^{\lfloor t/2 \rfloor} 2^k G \sum_{m=0}^{\infty} \frac{((-1)^t)^m}{(2m+1)^2}) = st \sum_{n=0,\ (2n+1)\nmid t}^{\infty} \frac{(-1)^n}{(2n+1)^2} + ((-1)^{\lfloor t/2 \rfloor} 2^k G^2).$$

In other words, we obtain the following quadratic equation for G:

$$G^2-(-1)^{\lfloor t/2\rfloor}\frac{st}{2^k}G+(-1)^{\lfloor t/2\rfloor}\frac{st}{2^k}\sum_{n=0,\ (2n+1)\nmid t}^{\infty}\frac{(-1)^n}{(2n+1)^2}.$$

The last is equal to

$$G^2-(-1)^{\lfloor t/2\rfloor}\frac{st}{2^k}G+(-1)^{\lfloor t/2\rfloor}t^2G\sum_{n=0,\ (2n+1)\nmid t}^{\infty}\frac{(-1)^n}{(2n+1)^2}.$$

Since $G\neq 0$, we have the next equation

$$G=(-1)^{\lfloor t/2\rfloor}\frac{st}{2^k}-(-1)^{\lfloor t/2\rfloor}t^2\sum_{n=0,\ (2n+1)\nmid t}^{\infty}\frac{(-1)^n}{(2n+1)^2}.$$

Indeed, we have

$$G=(-1)^{\lfloor t/2\rfloor}\frac{st}{2^k}-(-1)^{\lfloor t/2\rfloor}t^2(G+\epsilon),$$
$$G=(-1)^{\lfloor t/2\rfloor}t^2G-(-1)^{\lfloor t/2\rfloor}t^2(G+\epsilon),$$
$$G=-(-1)^{\lfloor t/2\rfloor}t^2\epsilon,$$

where

$$\epsilon=-\sum_{m=0}^{\infty}\frac{(-1)^{mt+\lfloor t/2\rfloor}}{t^2(2m+1)^2}=-(-1)^{\lfloor t/2\rfloor}\frac{G}{t^2}.$$

According to the above, we consider the following quadratic equation for t:

$$G=(-1)^{\lfloor t/2\rfloor}\frac{st}{2^k}-(-1)^{\lfloor t/2\rfloor}t^2(G+\epsilon),$$

$$t^2-\frac{s}{2^k(G+\epsilon)}t+(-1)^{\lfloor t/2\rfloor}\frac{G}{(G+\epsilon)}=0.$$

Since $\frac{s}{2^k(G+\epsilon)}>0$ due to $t>1$ (G can not be $\frac{s}{2^k}$ for natural s,k: it goes around with the representation $\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}$ and, for example, we can apply the above idea for s, which can be only **odd** in this case; note that G is definitely not $\frac{1}{2^k}$), we get

$$t=\frac{s}{2^{k+1}(G+\epsilon)}(1\pm\sqrt{1-\frac{4(-1)^{\lfloor t/2\rfloor}G(G+\epsilon)^2 2^{2k}}{(G+\epsilon)s^2}})=$$

$$=\frac{s}{2^{k+1}(G+\epsilon)}(1\pm\sqrt{1-\frac{(-1)^{\lfloor t/2\rfloor}G(G+\epsilon)2^{2k+2}}{s^2}}).$$

Using the Taylor series of $\sqrt{1+x}$ ($\frac{G(G+\epsilon)2^{2k+2}}{s^2}=\frac{4}{t^2}(1-(-1)^{\lfloor t/2\rfloor}\frac{1}{t^2})\leq\frac{8}{t^2}\leq\frac{8}{3^2}<1$), we come to

$$t_+\cong\frac{s}{2^k(G+\epsilon)}-\frac{(-1)^{\lfloor t/2\rfloor}G2^k}{s}-\frac{G^2(G+\epsilon)2^{3k}}{s^3},\ t_-\cong\frac{(-1)^{\lfloor t/2\rfloor}G2^k}{s}+\frac{G^2(G+\epsilon)2^{3k}}{s^3},$$

where t_- is impossible as $G=\frac{s}{2^kt}$ and $t\geq 3$.

Substituting $G=\frac{s}{2^kt_+}$, we derive

$$t_+\cong\frac{s}{2^k(G+\epsilon)}-\frac{(-1)^{\lfloor t_+/2\rfloor}G2^k}{s}-\frac{G^2(G+\epsilon)2^{3k}}{s^3}=\frac{t_+}{(1-(-1)^{\lfloor t_+/2\rfloor}\frac{1}{t_+^2})}-\frac{(-1)^{\lfloor t_+/2\rfloor}}{t_+}-\frac{(1-(-1)^{\lfloor t_+/2\rfloor}\frac{1}{t_+^2})}{t_+^3}=$$

$$=\frac{t_+}{(1-(-1)^{\lfloor t_+/2\rfloor}\frac{1}{t_+^2})}-\frac{(-1)^{\lfloor t_+/2\rfloor}}{t_+}-\frac{1}{t_+^3}+\frac{(-1)^{\lfloor t_+/2\rfloor}}{t_+^5}.$$

According to the above, we consider the following equation for $t_+ \neq 0$:

$$\frac{1}{(1-(-1)^{\lfloor t_+/2 \rfloor}\frac{1}{t_+^2})} - \frac{(-1)^{\lfloor t_+/2 \rfloor}}{t_+^2} - \frac{1}{t_+^4} + \frac{(-1)^{\lfloor t_+/2 \rfloor}}{t_+^6} \cong 1.$$

Despite the correct mien of the above expression, note that $1/(1-x)$ and $\sqrt{1-x}$ are different as series. Indeed, for $\sqrt{1-x}$ we have

$$1 - x/2 - x^2/8 - x^3/16 - (\mathbf{5}x^4)/128 - (\mathbf{7}x^5)/256 + O(x^6).$$

Hence, the acquired identity for t_+ can not be fulfilled. **Q.E.D.**

Remark 1. *There exists the following integration*

$$\int_0^{\infty} \frac{1}{1+x^2}\cos(kx)dx = \frac{\pi}{2}e^{-k}.$$

One way to see it is via the Fourier inversion theorem: we know that the Fourier transform of a function has a unique inverse. This carries over to the cosine transform as well. Moreover, the unique continuous function on the positive real axis with Fourier transform $\frac{1}{1+x^2}$ *is* e^{-k}.

Notice that if

$$I_n = \int \frac{x^n}{1+x^2}dx,$$

then

$$I_{n+2} + I_n = \frac{x^n}{n+1} + C.$$

Remark 2. *Are all* $\{1, {}^n\pi \mid n \in \mathbb{N}\}$ *linearly independent over* $\mathbb{Q}$, *where* ${}^n x$ *is tetration? Meaning none of exponents is an integer (we have not known that* $\pi^{\pi^{\pi^{\pi}}}$ *(56 digits) is not an integer).*

Moreover, at least one of e^e *and* e^{e^2} *must be transcendental due to W. D. Brownawell.*

Remark 3. *Is* $e + \pi$ *irrational?*

Note that $(x-e)(x-\pi) = x^2 - (e+\pi)x + e\pi$. *So, at least one of the coefficients* $e+\pi$, $e\pi$ *must be irrational.*

Remark 4. *Is* $\ln(\pi)$ *irrational?*

There exists such representation

$$\frac{\sin(x)}{x} = \prod_{n=1}^{\infty}(1 - \frac{x^2}{n^2\pi^2}).$$

Let $x = \frac{\pi}{2}$ *and then we have the Wallis product formulae for* $\frac{\pi}{2}$:

$$\frac{\pi}{2} = \prod_{n=1}^{\infty} \frac{2n}{2n-1}\frac{2n}{2n+1}.$$

Taking logarithms of this, we come to

$$\ln(\pi) = \ln(2) + \sum_{n=1}^{\infty}(2\ln(2n) - \ln(2n-1) - \ln(2n+1)).$$

Remark 5. *Is the Euler–Mascheroni constant* γ *irrational?*

$$\gamma = \lim_{n\to\infty}\left(\sum_{m=1}^{n}\frac{1}{m} - \log(n)\right).$$

Remark 6. *Is the Khinchin's constan* K_0 *irrational?*

$$K_0 = \prod_{n=1}^{\infty}(1 + \frac{1}{n(n+2)})^{\log_2 n}.$$

References

[1] Ivan Morton Niven, *Numbers: Rational and Irrational*, Mathematical Association of America, Year: 1961.