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# ON CONSTITUTIVE CHOICES FOR GROWTH TERMS IN BINARY FLUID MIXTURES

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**Abstract.** *The differential system of balance equations for a binary mixture of fluids as the system of a single fluid, plus that relating to diffusive motion, is used to present constitutive proposals for the terms of exchange and to verify the metaphysical principles of Truesdell. After two particular examples are considered presenting peculiar constitutive relations for such growth terms and study small plane waves in order to verify the influence of these terms on the solutions.*

## 1 INTRODUCTION

In the present paper we present the thermodynamic system of balance equations for a binary mixture of fluids such as the system of a single fluid, in addition to that pertinent to the relative motion, using a mere change of variables [1]. Next, we follow [15] to recognize all the thermo-mechanical fields as defined in the motion of the mixture as a whole and in the diffusive motion, so as to verify the three metaphysical principles of [16] and, further, we assume that the entropy flux is not equal to the heat flux divided by the temperature [9].

Furthermore, we recover the kinetic energy theorem for the binary mixture directly from the mechanical equations in order to propose the expressions of growths for the body by minimizing the production or destruction of mass and linear momentum: the different proposals take into account the essential features of classical models of two-phase mixture of fluids in the presence of production terms capable of describing the effects of diffusion, virtual inertia, inviscid inertial drag, and so on (see, e.g., [2, 4, 7]).

Finally, we consider two examples, such as the superfluid helium and a mixture of Euler fluids, and study the propagation of small vibrations near an undisturbed state of equilibrium for the mixture to verify the influence of the growth terms on the linear solutions.

## 2 BALANCE AXIOMS FOR BINARY MIXTURES OF FLUIDS

Let us consider the thermodynamic of a binary homogeneous mixture of fluids, where we assume that physical transfers, and eventual chemical reactions, are exchanges rather than true processes of creation or destruction, therefore we allow mass, linear momentum, rotational momentum, and energy of any constituents to change form, but do not allow the total mixture to produce these quantities, in agreement with the third metaphysical principle of Truesdell [16] that postulates, for a simple fluid mixture, the same balance laws as for a single fluid.

In this case, following [5, 14], it is possible to rewrite the usual differential system of balance laws in terms of average displacement and relative one (or drifts), rather than peculiar ones. Let us report here the balances of mass, linear momentum, moment of momentum and energy of the whole mixture with a single temperature $\theta$ for the constituents, respectively:

$$\dot{\rho} + \rho\,\mathrm{div} v = 0, \quad \rho\,\dot{v} = \mathrm{div}\,T + \rho\,b, \quad T = T^T,$$
$$\rho\dot{\epsilon} = \mathrm{div}\,h + \rho\,\lambda + T\cdot\mathrm{grad}\,v, \tag{1}$$

where the dot $\dot{}$ denotes the material derivative with respect to time following the mean motion $v$. In equations (1) we introduced the following quantities:

1) the density and the velocity of the mixture defined by

$$\rho := \rho_1 + \rho_2 \quad \text{and} \quad v := \mu\,v_1 + (1-\mu)\,v_2, \tag{2}$$

respectively, where

$$\mu := \frac{\rho_1}{\rho} \tag{3}$$

is the concentration of the first constituent: here and henceforth, the subscripts 1 and 2 stands for the first component of the mixture and the second one, respectively;

2) the Cauchy's stress tensor $T$ and the density per unit mass of body force $b$ of the mixture defined by

$$T := T_I - \rho\,\mu\,(1-\mu)\,w\otimes w \quad \text{and} \quad b := \mu\,b_1 + (1-\mu)\,b_2, \tag{4}$$

where $T_I := T_1 + T_2$ is the symmetric inner part of the stress and $w := v_2 - v_1$ the relative velocity: it should be noted that, in general, the peculiar stress tensors $T_j$ are not symmetric, but their sum $T_I$ must be for equations $(1)_1$ and (4);

3) the density per unit mass of total internal energy $\epsilon$ and of heating supply $\lambda$ of the mixture defined by

$$\epsilon := \epsilon_I + \frac{1}{2}\mu(1-\mu)\,w^2 \quad \text{and} \quad \lambda := \lambda_I + \mu(1-\mu)\,c\cdot w, \tag{5}$$

where the inner part are defined as $\epsilon_I := \mu\epsilon_1 + (1-\mu)\,\epsilon_2$ and $\lambda_I := \mu\,\lambda_1 + (1-\mu)\,\lambda_2$, while $c := b_2 - b_1$ is the drift body force;

4) the mixture heat flux vector

$$h := h_1 + h_2 + S^T w + \rho\,\mu\,(1-\mu)\left[\zeta + \frac{1}{2}(1-2\mu)w^2\right] w, \tag{6}$$

where $S := \mu\,T_2 - (1-\mu)\,T_1$ and $\zeta := \epsilon_2 - \epsilon_1$ are the drift stress and internal energy, respectively.

In addition, we need the drift balance equations of mass, linear momentum and moment of momentum, respectively, to close the differential system, that is:

$$\rho\,\dot{\mu} = \operatorname{div}\left[\rho\mu(1-\mu)w\right] + \rho\,\alpha, \qquad S - S^T = \rho M, \tag{7}$$

$$\begin{aligned}&\rho\,\mu\,(1-\mu)\left\{\dot{w} + \operatorname{grad}\,[v-(1-2\mu)w]\,w\right\} = \\ &\quad = \operatorname{div} S + \rho\,\mu\,(1-\mu)c - T\operatorname{grad}\mu + \rho\left\{\alpha\,[v-(1-2\mu)w] - m\right\},\end{aligned} \tag{8}$$

where $\rho\alpha$, $\rho m$ and $\rho M$ are the growths per unit volume of mass and linear and rotational momenta, respectively.

Furthermore, the entropy principle is expressed in the following form [9]:

$$\rho\,\dot{\eta} \geq \operatorname{div} k + \rho\,\lambda_I\,\theta^{-1}, \tag{9}$$

where $\eta := \mu\,\eta_1 + (1-\mu)\,\eta_2$ is the total entropy density per unit mass of the mixture and $k$ the mixture entropy flux vector defined by

$$k := \theta^{-1}\left\{h - S^T w - \rho\,\mu\,(1-\mu)\left[\zeta + \frac{1}{2}(1-2\mu)w^2 - \theta\xi\right] w\right\}, \tag{10}$$

where $\xi := \eta_2 - \eta_1$ is the drift entropy density; therefore, the classical proportionality relation between the entropy flux $k$ and the heat flow $h$ through the temperature is now modified as reported in the equation (6), where the thermal quantity $[-\rho\,\mu\,(1-\mu)\,(\theta^{-1}\zeta - \xi)\,w]$, as well as a term proportional to the interstitial power $\left\{-\theta^{-1}\left[\frac{1}{2}\rho\,\mu\,(1-\mu)(1-2\mu)w\otimes w + S\right]^T w\right\}$, affect the entropic flow.

Moreover, we should consider other two equations for $\zeta$ and $\xi$, but we supposed that the energy exchange between the constituents is so efficient that the mixture can be characterized by a single temperature; therefore, such an assumption is tantamount to restricting considerations to the mixture energy equation $(1)_4$ only, rather than to each energy equation separately [10], because the dissipation principle does not restrict the peculiar growths of energy: we need then work only with the mixture forms for reduced energy and entropy balances (see, also, [16]).

Besides, the rate of total radiant heating $\lambda$ differs from its inner part $\lambda_I$ only if the mass forces $b_j$ are not equal, and so $c \neq 0$ (see the relation (5)): in this case, the balance equation for total internal energy $(1)_4$ and the dissipation axiom for mixtures (9) show that the work of mass forces against the diffusion, although it provides warming to the body, does not contribute to the production of entropy.

## 3 'VIS VIVA' THEOREM AND CONSTITUTIVE PROPOSALS FOR EXCHANGE TERMS

Now we infer the kinetic energy theorem for the binary mixture, described in the previous section, directly from the linear momentum equations $(1)_2$ and (8) taking the scalar product of both sides by $v$ and $w$, respectively, using the mass equations $(1)_1$ and $(7)_1$ and the symmetry of $T$ $(1)_3$, integrating by parts, where possible, both sides over a domain $\mathcal{B}$ that moves with the mean velocity $v$ and summing at the end term by term; finally, we have the theorem:

$$
\begin{aligned}
&\frac{d}{d\tau}\int_{\mathcal{B}} \frac{1}{2}\rho\left[v^2+\mu\,(1-\mu)w^2\right]+\int_{\partial\mathcal{B}} \frac{1}{2}\rho\,\mu\,(1-\mu)\{[2\,v-(1-2\mu)w]\otimes w\}\cdot w\otimes n=\\
&=\int_{\mathcal{B}} \rho\left\{b\cdot v+\mu\,(1-\mu)\,c\cdot w+\left[m-\frac{1}{2}\alpha[2\,v-(1-2\mu)w]\right]\cdot w\right\}+\\
&+\int_{\partial\mathcal{B}}(v\cdot T_I n+w\cdot Sn)-\int_{\mathcal{B}}[T_I\cdot(\operatorname{grad}v-w\otimes\operatorname{grad}\nu)+S\cdot\operatorname{grad}w],
\end{aligned}
\tag{11}
$$

where $n$ is the unit vector of the exterior normal to the boundary $\partial\mathcal{B}$ of $\mathcal{B}$.

Therefore, the expressions of growths $\alpha$ and $m$ are of fundamental importance in the balance of the mechanical energy. In particular, the theorem (11) suggests that it is certainly not modified by the production or destruction of mass and quantity of motion if we choose $m$ equal to the contribution due to the creation of mass $\alpha$ with the unweighted average speed of components, *i.e.*, in this context,

$$
m=\alpha\left[v-\left(2^{-1}-\mu\right)w\right]\left\{=2^{-1}\alpha(v_1+v_2)\right\},
\tag{12}
$$

for which the corresponding variation of kinetic energy is exactly compensated by the power supplied [5]; we observe that relations $v_1=v-(1-\mu)w$ and $v_2=v+\mu w$ hold.

For a binary mixture of Euler fluids, we can choose a suitable expression for $m$, assuming the cancellation of $\alpha$ [14]:

$$
m=-\psi\mu\,(1-\mu)w\quad\text{and}\quad\alpha=0,
\tag{13}
$$

with $\psi$ a production term due to the interchange of momentum between the two species, for which the first integral on the right hand side of (11) becomes

$$
\int_{\mathcal{B}}\rho\,\{b\cdot v+\mu\,(1-\mu)\,\,(c-\psi w)\cdot w\}
$$

and the corresponding variation of kinetic energy due to the exchange of linear momentum behaves as a modification of the power of external actions only due to the relative motion.

Obviously, the choices are certainly a constitutive question and Müller [10, 11] proposed to take the sum of a quantity of the type $\alpha q$, with $q$ a velocity to be specified, plus an objective term $\tilde{m}$. This proposal is therefore certainly respected by (12) with $\tilde{m}$ vanishing. More in general, the author in [1] suggested to choose

$$
m=\alpha\left[v-\left(\frac{1}{2}-\mu\right)w\right]+\tilde{m};
\tag{14}
$$

this choice includes, in addition to (12), also the most common case that occurs in the works on superfluid helium, in which it is set

$$
m=\alpha(v+\mu w),
\tag{15}
$$

that corresponds to assume $\tilde{m} = \frac{1}{2}\alpha w$ (see, *e.g.*, [12] or equation $(6.180)_8$ of [11], where the helium is considered as a particular simple binary mixture).

Finally, we wish to present a further expression for $m$ in contrast to the previous proposals, and which concerns an immiscible mixture, the bubbly liquid: in fact, it shows a non-classical expression for its kinetic energy due to the presence of terms, in addition to the translational ones, which take into account the effects of virtual translation inertia, representing local non-uniformities in the flow of constituents when they move relative to each other, and local micro-variations of the volume of bubbles that move in the liquid environment, which is set in motion, giving origin to effects of expansional inertia. We do not present here those expressions, which are analytically calculated in the conservative case in [6], nor the relative kinetic energy theorem, but only precise that now, in absence of mass production ($\alpha = 0$), the exchange term between phases can be proportional also to the accelerations, differently from the proposal (14), *i.e.*,

$$m = -\varsigma \dot{w} - \bar{\varsigma}(\operatorname{grad} w)^T w, \tag{16}$$

with coefficients depending on $\rho$ and $\mu$; moreover, another term of inviscid drag due to inertia forces can be present and related to convective derivatives, following the liquid velocity, of the bubbles radius and of the bulk mass density of the gas, as hypothesized in [3] during experimental observations of the motion of bubbly fluids in a circular pipe.

## 4 SMALL PLANE VIBRATIONS IN BINARY MIXTURES

We want to test the constitutive expressions for growths, proposed in the previous section, in a simple wave propagation problem in superfluid helium viewed as a binary mixture whose superfluid component is not viscous and has no dynamic interactions with the normal component, and in a two-phase mixture of Euler fluids, *i.e.* fluids that are neither viscous nor heat conducting, and do not react chemically.

### 4.1 Superfluid Helium

In this first example we consider only mechanical waves and it is important to observe that the distinction, in the superfluid helium, of two kinematically separated components is only of a formal nature and, therefore, it seems difficult to imagine that on them there are distinct actions at a distance that can be arbitrarily assigned, as do in [8]; here we will suppose, thus, that

$$b_1 = b_2 = b, \qquad \lambda_1 = \lambda_2 \quad \Rightarrow \quad c = 0, \qquad \lambda = \lambda_I = \lambda_1 = \lambda_2, \tag{17}$$

for definitions $(4)_2$ and $(5)_2$. Furthermore, we must choose constitutive expressions for growth terms: for $m$ we use the expression $(12)_1$, while, for $\alpha$, we suppose a dependency on mass density rates $\dot{\rho}$ and $\dot{\mu}$:

$$\rho\alpha = \beta\dot{\rho} + \gamma\dot{\mu} \quad (\beta, \gamma \,\text{material costants}); \tag{18}$$

at the end, we impose to stress tensors $T_I$ and $S$ to be isotropic, as for ideal fluids, therefore we have:

$$T = -\left[\pi I + \rho\,\mu\,(1-\mu)\,w \otimes w\right], \qquad S = -\chi I, \tag{19}$$

where the pressures $\pi$ and $\chi$ are scalar functions depending on $\rho$ and $\mu$.

**Remark.** Regarding some suggestions from statistical mechanics for eventual thermal developments [11], we observe that the specific entropy density for the superfluid component of the mixture and the internal drifting energy are usually supposed to vanish:

$$\eta_2 = 0, \quad \epsilon_1 = \epsilon_2 \quad \Rightarrow \quad \eta = \mu\,\eta_1, \quad \xi = -\eta_1, \quad \zeta = 0. \tag{20}$$

Finally, for our purposes, the system of balance equations consists of $(1)_{1,2}$, $(7)_1$ and (8) putting the total and relative mass forces $b$ and $c$, respectively, at zero and replacing the constitutive laws $(12)_1$, (18) and (19):

$$\dot{\rho} + \rho\,\mathrm{div} v = 0, \qquad (\rho-\gamma)\,\dot{\mu} = \mathrm{div}\,[\rho\mu(1-\mu)w] + \beta\dot{\rho}, \tag{21}$$

$$\rho\,\dot{v} = -\pi_\rho\,\mathrm{grad}\,\rho - \pi_\mu\,\mathrm{grad}\,\mu - \mathrm{div}\,\left[\rho\,\mu\,(1-\mu)\,w\otimes w\right], \tag{22}$$

$$\begin{aligned}\rho\,\mu\,(1-\mu)\,\{\dot{w} + \mathrm{grad}\,[v-(1-2\mu)w]\,w\} = -\chi_\rho\,\mathrm{grad}\,\rho - \chi_\mu\,\mathrm{grad}\,\mu + \\ +\pi(\mathrm{grad}\,\mu) + \rho\,\mu\,(1-\mu)[w\cdot(\mathrm{grad}\,\mu)]\,w - \left(\frac{1}{2}-\mu\right)(\beta\dot{\rho}+\gamma\dot{\mu})w,\end{aligned} \tag{23}$$

where we used the dependency of pressures on $\rho$ and $\mu$ and indicated partial derivatives as, *e.g.*, $\pi_\rho = \left(\frac{\partial\pi}{\partial\rho}\right)_\mu$, and so on.

Now we observe that there is a constant solution $\nu_0 = (\rho_0, \mu_0, v_0, w_0)$ of (21)-(23) and thus we can investigate on the propagation of perturbations of small-amplitude near an undisturbed state of equilibrium for the mixture characterized by:

$$\rho = \rho_0 + \tilde{\rho}, \quad \mu = \mu_0 + \tilde{\mu}, \quad v = v_0 + \tilde{v}, \quad w = w_0 + \tilde{w}, \tag{24}$$

where the quantities with the tilde will be thought small (for stability reasons); therefore, we linearize the system of balance equations around the state $\nu_0$, disregarding the non-linear terms with the tilde and in their partial derivatives, to obtain:

$$\frac{\partial\tilde{\rho}}{\partial\tau} + v_0\cdot\mathrm{grad}\,\tilde{\rho} + \rho_0\,\mathrm{div}\,\tilde{v} = 0, \tag{25}$$

$$\begin{aligned}(\rho_0-\gamma)\left(\frac{\partial\tilde{\mu}}{\partial\tau} + v_0\cdot\mathrm{grad}\,\tilde{\mu}\right) = \beta\left(\frac{\partial\tilde{\rho}}{\partial\tau} + v_0\cdot\mathrm{grad}\,\tilde{\rho}\right) - \\ -\mu_0(1-\mu_0)\,[w_0\cdot\mathrm{grad}\,\tilde{\rho} + \rho_0\mathrm{div}\,\tilde{w}] - \rho_0(1-2\mu_0)w_0\cdot\mathrm{grad}\,\tilde{\mu},\end{aligned} \tag{26}$$

$$\begin{aligned}\rho_0\left[\frac{\partial\tilde{v}}{\partial\tau} + (\mathrm{grad}\,\tilde{v})v_0\right] = -\left(\pi_\rho\right)_0\mathrm{grad}\tilde{\rho} - \left(\pi_\mu\right)_0\mathrm{grad}\tilde{\mu} - \mu_0(1-\mu_0)(w_0\cdot\mathrm{grad}\tilde{\rho})w_0 - \\ -\rho_0(1-2\mu_0)(w_0\cdot\mathrm{grad}\,\tilde{\mu})w_0 - \rho_0\mu_0(1-\mu_0)\left[(\mathrm{grad}\,\tilde{w})w_0 + w_0\mathrm{div}\,\tilde{w}\right],\end{aligned} \tag{27}$$

$$\begin{aligned}\rho_0\mu_0(1-\mu_0)\left\{\frac{\partial\tilde{w}}{\partial\tau} + (\mathrm{grad}\,\tilde{w})v_0 + [\mathrm{grad}\,\tilde{v} + (1-2\mu_0)\mathrm{grad}\,\tilde{w} - w_0\otimes\mathrm{grad}\,\tilde{\mu}]\,w_0\right\} = \\ = \left(\mu_0 - \frac{1}{2}\right)\left[\beta\left(\frac{\partial\tilde{\rho}}{\partial\tau} + v_0\cdot\mathrm{grad}\tilde{\rho}\right) + \gamma\left(\frac{\partial\tilde{\mu}}{\partial\tau} + v_0\cdot\mathrm{grad}\tilde{\mu}\right)\right]w_0 - \\ -\left(\chi_\rho\right)_0\mathrm{grad}\,\tilde{\rho} - \left[\left(\chi_\mu\right)_0 - \pi_0\right]\mathrm{grad}\,\tilde{\mu},\end{aligned} \tag{28}$$

where we introduced the perturbations $\pi = \pi_0 + \tilde{\pi}$ and $\chi = \chi_0 + \tilde{\chi}$ for the pressures and calculate quantities with the subscript $_0$ in $\nu_0$.

We look for plane wave solutions of the form:

$$\tilde{\rho} = \delta\rho\;\phi(x,\tau), \quad \tilde{\mu} = \delta\mu\;\phi(x,\tau), \quad \tilde{v} = \delta v\;\phi(x,\tau), \quad \tilde{w} = \delta w\;\phi(x,\tau), \tag{29}$$

where $\delta\rho$ and $\delta\mu$ are scalar wave amplitudes, while $\delta v$ and $\delta w$ are vector wave amplitudes; moreover,

$$\phi(x,\tau) = \exp[i(k\cdot x - \omega\tau)], \tag{30}$$

with $k$ and $\omega$ are the wave vector and the frequency, respectively. Let us denote with $\hat{n}$ the unit vector normal to the wave front and with $v_f$ the phase velocity, such that

$$k = \kappa\hat{n}, \quad v_f = \frac{\omega}{\kappa}, \tag{31}$$

where $\kappa$ is the length of $k$, and, besides, with a subscript $_n$ the normal component to the wave of vectors, *e.g.*, $v_n = v\cdot\hat{n}$.

By replacing the waves (29) in (25)-(28) and operating the partial derivatives, we obtain the following homogeneous linear algebraic system resulting:

$$(v_{0n} - v_f)\,\delta\rho + \rho_0\,\delta v_n = 0, \tag{32}$$
$$-\beta\,(v_{0n} - v_f)\,\delta\rho + (\rho_0 - \gamma)\,(v_{0n} - v_f)\,\delta\mu + \rho_0\mu_0\,(1-\mu_0)\,\delta w_n = 0, \tag{33}$$
$$\left[(\pi_\rho)_0\,\delta\rho + (\pi_\mu)_0\,\delta\mu\right]\hat{n} + \rho_0\,(v_{0n} - v_f)\,\delta v = 0, \tag{34}$$
$$\left[(\chi_\rho)_0\,\delta\rho + \left((\chi_\mu)_0 - \pi_0\right)\delta\mu\right]\hat{n} + \rho_0\mu_0\,(1-\mu_0)\,(v_{0n} - v_f)\,\delta w = 0, \tag{35}$$

where we supposed that, in the undisturbed state, the velocity is the same for all components, *i.e.*, $w_0 = 0$.

Therefore, we have two different classes of waves:

1) **Material waves.** If the phase velocity $v_f = v_{0n}$, then we have

$$\delta v_n = \delta w_n = 0 \tag{36}$$

from equations (32)-(33) and

$$\delta\pi = (\pi_\rho)_0\,\delta\rho + (\pi_\mu)_0\,\delta\mu = 0, \quad (\chi_\rho)_0\,\delta\rho + \left[(\chi_\mu)_0 - \pi_0\right]\delta\mu = 0 \tag{37}$$

from equations (34)-(35); this system has the non-trivial solution

$$\delta\rho = -\frac{(\pi_\mu)_0}{(\pi_\rho)_0}\delta\mu \tag{38}$$

only if

$$(\pi_\rho)_0\left[(\chi_\mu)_0 - \pi_0\right] = (\chi_\rho)_0\,(\pi_\mu)_0 \quad \text{with} \quad (\pi_\rho)_0 > 0. \tag{39}$$

The wave propagates with phase velocity equal to the normal component of the undisturbed speed $v_f = v_{0n}$, and the vector wave amplitudes $\delta v$ e $\delta w$ are tangent to the plane of the wave, therefore we have two *transverse* waves of velocities, one mean and the other relative, and a mass wave of amplitude (38) associated with that of concentration, while the material wave does not carry perturbations of the total pressure, but only of drift pressure given by

$$\delta\chi = -\pi_0\delta\mu. \tag{40}$$

2) **Acoustic waves.** If the phase velocity $v_f \neq v_{0n}$, then we have from equations (34)-(35) that $\delta v$ and $\delta w$ are parallel to $\hat{n}$, so the wave is *longitudinal* and, by multiplying each of them

scalarly by $\hat{n}$, we obtain the homogeneous system in $\delta\rho, \delta\mu, \delta v_n$ and $\delta w_n$:

$$(v_{0n} - v_f)\,\delta\rho + \rho_0\,\delta v_n = 0, \tag{41}$$
$$-\beta\,(v_{0n} - v_f)\,\delta\rho + (\rho_0 - \gamma)\,(v_{0n} - v_f)\,\delta\mu + \rho_0\mu_0\,(1 - \mu_0)\,\delta w_n = 0, \tag{42}$$
$$(\pi_\rho)_0\,\delta\rho + (\pi_\mu)_0\,\delta\mu + \rho_0\,(v_{0n} - v_f)\,\delta v_n = 0, \tag{43}$$
$$(\chi_\rho)_0\,\delta\rho + \left[(\chi_\mu)_0 - \pi_0\right]\delta\mu + \rho_0\mu_0\,(1 - \mu_0)\,(v_{0n} - v_f)\,\delta w_n = 0. \tag{44}$$

From equations (41)-(43) we obtain

$$\delta v_n = -\rho_0^{-1}\,(v_{0n} - v_f)\,\delta\rho, \tag{45}$$
$$\delta\mu = (\pi_\mu)_0^{-1}\left[(v_{0n} - v_f)^2 - (\pi_\rho)_0\right]\delta\rho, \tag{46}$$
$$\delta w_n = \left\{\beta + \frac{(\gamma - \rho_0)}{(\pi_\mu)_0}\left[(v_{0n} - v_f)^2 - (\pi_\rho)_0\right]\right\}\frac{(v_{0n} - v_f)}{\rho_0\mu_0\,(1 - \mu_0)}\delta\rho, \tag{47}$$

while the (44) has real non-trivial solutions for $\delta\rho$ only if

$$(v_f - v_{0n})^2 = \frac{-b \pm \sqrt{\Delta}}{2a} > 0, \tag{48}$$

with

$$a = (\gamma - \rho_0) \neq 0, \quad b = \beta\,(\pi_\mu)_0 - (\gamma - \rho_0)\,(\pi_\rho)_0 + \left[(\chi_\mu)_0 - \pi_0\right],$$
$$c = (\chi_\rho)_0\,(\pi_\mu)_0 - (\pi_\rho)_0\left[(\chi_\mu)_0 - \pi_0\right] \quad \text{and} \quad \Delta = b^2 - 4ac \geq 0. \tag{49}$$

Therefore, there are four waves moving with respect to the superfluid with speeds, respectively, $\pm\sqrt{\frac{-b\pm\sqrt{\Delta}}{2a}}$; such sound waves are all longitudinal, and are characterized by variations of total mass density, as well as of concentration, mean and relative velocities, other than pressures, whose amplitudes are related to the total mass one and given by equations (45)-(46), respectively.

The influence of the coefficients $\beta$ and $\gamma$ of the mass exchange term $\alpha$ on the longitudinal phase velocities, as well as on the amplitude of the relative velocity perturbation, is evident in (48) and (47), respectively, and so the response of the mixture is modified by $\alpha$.

Finally, we observe that, if we impose to the normal component to behave as an isothermal perfect gas with pressure $p_1 = \sigma\rho$, while $p_2 = 0$ and the constant $\sigma := Bm_1^{-1}\theta_0$, where $m_1$ is the atomic mass of the normal constituent, $B$ is the Boltzmann constant and $\theta_0$ the temperature in the unperturbed state, we have, from the expressions for $T_I$ and $S$, that

$$\pi := p_1 + p_2 = \sigma\rho \quad \text{and} \quad \chi := \mu p_2 - (1 - \mu)p_1 = -(1 - \mu)\sigma\rho. \tag{50}$$

In this case, $c = 0$ and $\Delta = (\rho_0 - \gamma)^2\sigma^2$ and only two solutions of (48) hold: $v_f = v_{0n} \pm \sqrt{\sigma}$, that is the velocity of sound wave of the mixture is like the perfect gas despite the growths of mass and linear momentum are different from zero.

### 4.2 Mixture of Euler fluids

In the second example we analyze the propagation of plane wave in a mixture of inviscid fluids, that do not conduct heat and do not react chemically, as introduced in [13], *i.e.*, a binary mixture of Euler fluids that obey the thermal equations of state of classical ideal gases for which the total pressure $\pi = R\rho\theta$, while $\chi = 0$, in constitutive laws (19) with $R$ a constant proportional to the Boltzmann constant $B$ (see, also, constitutive relations (2.9) and $(2.15)_{1,3}$ of [14]).

Moreover, we suppose null the body actions at distance $b$ and $c$ and the radiation $\lambda$, while, for the exchange terms, we use the proposal (16), with $\bar{\varsigma} = 0$, and $\alpha = 0$; thus, in addition to the mechanical equations of the previous subsection, we also use the energy balance $(1)_4$, and so the system of balance equations becomes

$$\begin{aligned}
&\dot{\rho} + \rho\,\mathrm{div} v = 0,\ \rho\,\dot{\mu} = \mathrm{div}\,[\rho\mu(1-\mu)w],\ \rho\,\dot{v} = -\mathrm{grad}\,\pi - \mathrm{div}\,\left[\rho\,\mu\,(1-\mu)\,w\otimes w\right],\\
&\rho\,\mu\,(1-\mu)\,\{\dot{w} + \mathrm{grad}\,\left[v-(1-2\mu)w\right]w\} = -\pi(\mathrm{grad}\,\mu) -\\
&\qquad\qquad -\rho\,\mu\,(1-\mu)[w\cdot(\mathrm{grad}\,\mu)]\,w + \rho\,\varsigma\,\dot{w},\\
&\rho\dot{\epsilon} = \mathrm{div}\,h - \pi\,\mathrm{div}\,v - \rho\,\mu\,(1-\mu)\,[(\mathrm{grad}\,v)w]\cdot w;
\end{aligned} \tag{51}$$

here we observe that, for definition (6), the heat flux vector of the mixture $h$ is different from zero even if the individual components do not conduct heat and $S = 0$.

The inner part and the drift of internal energy are given in [14] by:

$$\epsilon_I = \hat{\Gamma}(\mu)R\theta,\ \ \zeta := \Gamma R\theta, \quad \text{with}\ \hat{\Gamma} = \frac{\mu}{\gamma_1 - 1} - \frac{1-\mu}{\gamma_2 - 1},\ \Gamma = \frac{1}{\gamma_1 - 1} - \frac{1}{\gamma_2 - 1}, \tag{52}$$

respectively, where $\gamma_i$ is the ratio of the specific heat of the constituent $i = 1, 2$; therefore, from definitions $(5)_1$ and (6), we have:

$$\epsilon = \hat{\Gamma}R\theta + \frac{1}{2}\mu(1-\mu)w^2, \quad h := \rho\,\mu\,(1-\mu)\left[\Gamma R\theta + \frac{1}{2}(1-2\mu)w^2\right]w, \tag{53}$$

Now, in addition to perturbations (24) with $w_0 = 0$, we consider also a perturbation of the temperature

$$\theta = \theta_0 + \tilde{\theta} = \theta_0 + \delta\theta\,\phi(x,\tau), \tag{54}$$

with $\phi$ given by (30). Again, we linearize the system of balance equations around the unperturbed state, disregarding the non-linear 'tilde' terms, to have:

$$\begin{aligned}
&\frac{\partial\tilde{\rho}}{\partial\tau} + v_0\cdot\mathrm{grad}\,\tilde{\rho} + \rho_0\,\mathrm{div}\,\tilde{v} = 0,\\
&\rho_0\left(\frac{\partial\tilde{\mu}}{\partial\tau} + v_0\cdot\mathrm{grad}\,\tilde{\mu}\right) = -\rho_0\mu_0(1-\mu_0)\mathrm{div}\,\tilde{w},\\
&\rho_0\left[\frac{\partial\tilde{v}}{\partial\tau} + (\mathrm{grad}\,\tilde{v})v_0\right] = -R\left(\rho_0\,\mathrm{grad}\,\tilde{\theta} + \theta_0\,\mathrm{grad}\,\tilde{\rho}\right),\\
&\rho_0\left[\mu_0(1-\mu_0) - \varsigma\right]\left[\frac{\partial\tilde{w}}{\partial\tau} + (\mathrm{grad}\,\tilde{w})v_0\right] = -R\,\rho_0\,\theta_0\,\mathrm{grad}\,\tilde{\mu},\\
&\hat{\Gamma}_0\left(\frac{\partial\tilde{\theta}}{\partial\tau} + v_0\cdot\mathrm{grad}\,\tilde{\theta}\right) + \Gamma\theta_0\left(\frac{\partial\tilde{\mu}}{\partial\tau} + v_0\cdot\mathrm{grad}\,\tilde{\mu}\right) = \theta_0\left[\Gamma\mu_0(1-\mu_0)\mathrm{div}\,\tilde{w} - \mathrm{div}\,\tilde{v}\right]
\end{aligned} \tag{55}$$

Inserting the plane wave solutions (29) and $(54)_2$, we obtain the following algebraic system for amplitudes:

$$(v_{0n} - v_f)\,\delta\rho + \rho_0\,\delta v_n = 0, \tag{56}$$

$$(v_{0n} - v_f)\,\delta\mu + \mu_0\,(1-\mu_0)\,\delta w_n = 0, \tag{57}$$
$$R\,(\rho_0\delta\theta + \theta_0\delta\rho)\,\hat{n} + \rho_0\,(v_{0n} - v_f)\,\delta v = 0, \tag{58}$$
$$R\,\theta_0\,\delta\mu\,\hat{n} + \left[\mu_0(1-\mu_0) - \varsigma\kappa^{-1}\right](v_{0n} - v_f)\,\delta w = 0, \tag{59}$$
$$\hat{\Gamma}_0\,(v_{0n} - v_f)\,\delta\theta + \Gamma\theta_0\,(v_{0n} - v_f)\,\delta\mu + \theta_0\,[\delta v_n - \Gamma\mu_0(1-\mu_0)\delta w_n] = 0. \tag{60}$$

As in the previous example, we have two different classes of waves:

**1) Material waves.** If the phase velocity coincides with the normal component of the undisturbed speed, $v_f = v_{0n}$, then we have

$$\delta v_n = \delta w_n = \delta\mu = 0 \tag{61}$$

from equations (56), (57) and (59), moreover, from (58), we obtain

$$\delta\pi = R\,(\rho_0\delta\theta + \theta_0\delta\rho) = 0, \quad \text{or} \quad \delta\theta = -\rho_0^{-1}\theta_0\delta\rho; \tag{62}$$

the amplitude of the vector waves $\delta v$ e $\delta w$ are tangent to the plane of the wave, therefore, we still have two *transverse* waves of velocities, one mean and the other relative, and a thermal wave of amplitude $(62)_2$ associated with one of total mass, while the material wave does not carry perturbations of the total pressure, nor of concentration. There is no influence of the growth term of linear momentum $m$.

**2) Sound waves.** Instead, if the phase velocity $v_f \neq v_{0n}$, then we have two different solutions of the system (56)-(60), in fact, from equations (58) and (59) we deduce that $\delta v$ and $\delta w$ are parallel to $\hat{n}$, so the wave is *longitudinal* and, by multiplying each of them scalarly by $\hat{n}$, we can obtain the following homogeneous system in $\delta\rho, \delta\mu, \delta\theta, \delta v_n$ and $\delta w_n$:

$$\delta v_n = \rho_0^{-1}\,(v_f - v_{0n})\,\delta\rho, \qquad \delta w_n = [\mu_0\,(1-\mu_0)]^{-1}\,(v_f - v_{0n})\,\delta\mu, \tag{63}$$
$$R\delta\theta = \rho_0^{-1}\left[(v_f - v_{0n})^2 - R\theta_0\right]\delta\rho, \quad \left[R\,\theta_0 - (1-\bar{\varsigma})\,(v_f - v_{0n})^2\right]\delta\mu = 0, \tag{64}$$
$$\hat{\Gamma}_0\,\delta\theta + 2\Gamma\,\theta_0\,\delta\mu - \rho_0^{-1}\theta_0\,\delta\rho = 0, \tag{65}$$

with $\hat{\varsigma} = \varsigma\,[\mu_0(1-\mu_0)\kappa]^{-1}$, and so, for equations $(64)_2$, we must split it in two cases:

**2.a)** If $\delta\mu = 0$, thus $\delta w_n = 0$, $\delta\rho \neq 0$, and, therefore, we can substitute relation (65) in $(64)_1$ to get the two phase velocities

$$v_f = v_{0n} \pm c_1, \quad \text{with} \quad c_1^2 := R\theta_0\left(1 + \hat{\Gamma}_0^{-1}\right) > 0, \tag{66}$$

and amplitudes

$$\delta\theta = \left(\rho_0\,\hat{\Gamma}_0\right)^{-1}\theta_0\,\delta\rho, \quad \delta v_n = \pm\rho_0^{-1}c_1\delta\rho. \tag{67}$$

**2.b)** If $\delta\mu \neq 0$, from relation (64) we obtain the two phase velocities

$$v_f = v_{0n} \pm c_2, \quad \text{with} \quad c_2^2 := R\theta_0\,(1-\hat{\varsigma})^{-1} > 0, \tag{68}$$

and amplitudes

$$\delta\theta = \frac{\theta_0\hat{\varsigma}}{\rho_0(1-\hat{\varsigma})}\delta\rho, \quad \delta\mu = \left(1 - \frac{\hat{\Gamma}_0\,\hat{\varsigma}}{1-\hat{\varsigma}}\right)\frac{\delta\rho}{2\rho_0\Gamma},$$
$$\delta v_n = \pm\rho_0^{-1}c_2\delta\rho, \quad \delta w_n = \frac{\pm c_2}{2\rho_0\Gamma\,[\mu_0\,(1-\mu_0)]}\left(1 - \frac{\hat{\Gamma}_0\,\hat{\varsigma}}{1-\hat{\varsigma}}\right)\delta\rho. \tag{69}$$

Consequently, there are two types of longitudinal sound waves moving with respect to the mean motion of the Euler mixture and which present also thermal effects: in the first case, the velocities are, respectively, $c_1$ and $-c_1$ and such sound waves are marked by variations of total mass, as well as of temperature and average velocity, while do not carry perturbations of the concentration, nor of relative velocity. The speed $c_1$ clearly depends on the temperature $\theta_0$ and increases as it grows, like a single perfect gas; it also depends on the constituent ratios of the specific heat through the coefficient $\hat{\Gamma}_0$, while the non-zero amplitudes are given by (67).

In the second instance, velocities are, respectively, $c_2$ and $-c_2$ and such acoustic waves are characterized by perturbations of all quantities, whose amplitudes are given by equations (69). Now the wave speed $c_2$ depend not only on the temperature $\theta_0$, but also on the coefficient $\hat{\varsigma}$ of the exchange term of linear momentum $m$, therefore, only in these solutions, its expression significantly changes the response of the mixture to perturbations.

## 5 CONCLUDING REMARK

In this paper we have studied a binary mixture of fluids viewed as a single fluid with new fields describing the diffusion flux and the concentration of one constituent. In particular, we have discussed some constitutive choices for the terms of mass and linear momentum growths and verified the respect of Truesdell's metaphysical principles; moreover, we have examined small plane perturbations in a region where the mixture is in a state of rest (or uniform rectilinear motion) and without diffusion flux, in order to verify the influence of the production terms on the outcome.

The results we have obtained are in full agreement with the point of view put forward by the model; in particular, we analyzed an isothermal superfluid helium, seen as a binary mixture of a normal component and a superfluid component, and a mixture of Euler fluids.

In the first example, we have obtained the classical transverse solutions with phase velocity equal to the normal component of the undisturbed speed which does not carry perturbations of the total pressure; instead, the influence of the mass exchange term appears in the four phase velocities of the longitudinal sound waves, as well as in the amplitude of the perturbation of the relative speed.

Moreover, in the last example, for the transverse material solutions, we still have two classical waves, as for a single perfect gas, with a thermal wave associated with the mass one, but again the growth coefficient disappears; regarding the longitudinal sound waves, the first two solutions give small vibrations only in the mean motion of the whole mixture without growth influence, which, instead, clearly appear in the last two types of acoustic waves with non-null perturbations of all variables.

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