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New Existence Results for Odd Perfect Numbers of the Form n = p^α x N^2β

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NEW EXISTENCE RESULTS FOR ODD PERFECT NUMBERS OF THE FORM $n = p^{\alpha} N^{2\beta}$

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Abstract. Let n be an odd perfect number of the form $n = p^{\alpha} N^{2\beta}$, where N, β, α are positive integers, $N > 2$ is square free, p is a prime satisfying $p \equiv \alpha \equiv 1 \pmod 4$ and $p \nmid N$. We prove that $\beta \neq 27, 42$.

1. Introduction and Results

Let n be an odd positive integer and $\sigma(n)$ be the sum of all positive divisors of n. It is known that n is perfect if it satisfies $\sigma(n) = 2n$. Euler proved that an odd perfect number n is of the form $n = p^{\alpha} \prod_{i=1}^{k} p_i^{2\beta_i}$, where p_i, p are distinct primes and p is the special prime satisfying $p \equiv \alpha \equiv 1 \pmod 4$. Setting $\beta_i = \beta$ for all $i = 1, 2, \ldots, k$ and $N = \prod_{i=1}^{k} p_i$, we can write

$$n = p^{\alpha} N^{2\beta} \tag{1.1}$$

In this paper, we study odd perfect numbers of the form (1.1). Thus from now on without explicit mention, we consider n to be of the form (1.1). It has been proved that n cannot be perfect for the cases $\beta = 2, 3, 5, 6, 8, 11, 12, 14, 17, 18, 24, 62$ (See [2] for respective references). Although proving that n does not exist for any β remains an open problem, a few generalizations have been made. For example McDaniel [5] proved that we cannot have the case $\beta \equiv 1 \pmod 3$, while more recently, Yamada [7] proved that n has $2\beta^2 + 6\beta + 3$ or fewer prime divisors. Fletcher, Nielsen, and Ochem [6] proved that if n is an odd integer of the form $n = p^{\alpha} \prod_{i=1}^{k} p_i^{2\beta_i}$, where p_i, p are distinct primes and $2\beta_i + 1$ is divisible by 5, then n cannot be perfect. However, their proof needs a huge amount of calculation. In this paper, we show a weaker result in a more simple way and with a smaller amount of calculation. Building on the work of McDaniel [1], we prove the following theorem.

Theorem 1.1. *Let $n = p^{\alpha} N^{2\beta}$ be an odd perfect number, where N, β, α are positive integers, p is the special prime and $N > 2$ is square free. Then $\beta \neq 27, 42$.*

2. Preliminaries

Let n be a perfect number of the form (1.1), $\sigma(n)$ be the sum of positive divisors of n, q be an arbitrary prime and $\Phi_m(x)$ be the m^{th} cyclotomic polynomial of x. Then the following results hold.

(a) $\sigma(p_i^{2\beta}) = \prod_{\substack{d \mid (2\beta+1) \\ d > 1}} \Phi_d(p_i)$. Furthermore, if $r \mid \Phi_q(p_i)$, then either $r = q$ or $r \equiv 1 \pmod q$.

(See Theorem 94 and Theorem 95 in [4].)

(b) The special prime p satisfies $p \equiv 1$ or $5 \pmod{12}$. Furthermore, if $5|n$, then $p \equiv 1 \pmod{12}$. (See [1] Section 4.)

(c) It follows from (a) and (b) that if $q|n$ and $q \not\equiv 1$ or $5 \pmod{12}$, then $q^{2\beta}||n$ and consequently $\Phi_r(q)|n$ for any prime divisor r of $2\beta+1$.

(d) If $q^c||(2\beta+1)$ for some positive integer c, then at most $\frac{2\beta}{c}$ prime divisors of n are congruent $1 \pmod{q}$. (See [1] Section 2, Lemma 1.)

(e) If $q|(2\beta+1)$, then q divides n. (See Kanold [3].)

(f) If q divides $\frac{p+1}{2}$, then q divides n.

3. The case $\beta = 27$

Our proof proceeds by contradiction. We assume that an odd perfect number n exists for a particular β and use the factor chain argument to show that the number of possible prime factors generated contradicts (d).

Suppose that a perfect number n with $\beta = 27$ exists, then by (e), $5|n$ and it follows from (d) that n has at most 54 prime factors congruent $1 \pmod 5$. Since $5|n$, it follows from (c) that $\Phi_5(5)|n$, thus 11 and 71 are divisors of n. Since each of 11 and 71 do not qualify as a possible special prime, it follows from (c) that $\Phi_5(11)|n$ and $\Phi_5(71)|n$. Computing $\Phi_5(q)$ for some prime divisors q along the factor chain except when q is possibly a special prime, we obtain at least 55 primes (see Table 1 below) that are congruent $1 \pmod 5$ which is a contradiction.

4. The case $\beta = 42$

Suppose that a perfect number n with $\beta = 42$ exists, then by (e), $5|n$ and by (d), n has at most 85 prime factors congruent $1 \pmod 5$. Since $85|n$, it follows from (c) that $\Phi_5(5)|n$ and $\Phi_5(17)|n$. Starting the factor chain with $\Phi_5(5)$, we obtain 21 distinct prime factors (see Table 2 below) that are congruent $1 \pmod 5$. We obtain more prime factors by considering a factor chain that starts with $\Phi_5(17)$. Since 88741 divides $\Phi_5(17)$, it follows that 88741 divides n.

If 88741 is not a special prime, we obtain at least 87 primes (see Table 2 below) that are congruent $1 \pmod 5$ which is a contradiction.

If 88741 is special, it follows from (f) that 44371 divides n. Computing more primes along the factor chain that begins with $\Phi_5(44371)$, noting that no prime along this chain can be special, we obtain at least 65 primes (see Table 3 below) that are congruent $1 \pmod 5$ and are distinct from the 21 primes obtained from the factor chain starting with $\Phi_5(5)$. Altogether, we have 86 primes that are congruent $1 \pmod 5$ which is a contradiction.

Table 1. Table showing prime factors of $\Phi_5(q)$ and $\Phi_{11}(q)$ for some selected primes.

q	$\Phi_5(q)$	$\Phi_{11}(q)$
5	11, 71	
11	3221	15797
71	211, 2221	
211	1361	
1361	11831, 58044391	
11831	61, 3724261, 17249741	
17249741	31, 41, 1266549301470542329410701	
41	579281	
15797	17311991, 3597318971	
17311991	17471, 22571, 4141492142365002331	
17471	23860243716161	
22571	2671, 444971, 43676401	
2671	571, 147389551	
444971	78407535355557337126741	
571	1831, 11631811	
147389551	1547272999412852271559684070641	
1831	151, 43680671	
11631811	30257567468971582139834701	
31	17351	
151	104670301	
43680671	2531, 37924421, 7585358408296345091	
17351	1648012040336791	
2531	1721, 17971, 265471	
1721	10781, 43411	
17971	20861385065812741	
265471	181, 44501, 964151, 3119771	
10781	204331, 13224285131	
43411	6571, 9826865422541	
44501	1531, 403086353101	

Table 2. Table showing prime factors of $\Phi_5(q)$ for some selected primes.

q	$\Phi_5(q)$
5	11, 71
11	3221
71	211, 2221
211	1361
1361	11831, 478551301
11831	61, 3724261, 17249741
17249741	31, 41, 1266549301470542329410701
31	17351
41	579281
17351	1648012040336791
579281	2131, 1123759171, 9404401961
2131	4126364997061
17	88741
88741	4451, 5441, 46558947881
4451	2411, 32565724591
5441	601, 9409976951
2411	131, 2531, 20390861
131	973001
2531	1721, 17971, 265471
20390861	751, 254362896481589832203291
973001	12601, 19801, 23175534509471
1721	10781, 43411
17971	20861385065812741
265471	181, 44501, 964151, 3119771
751	46061, 125731
10781	204331, 13224285131
43411	6571, 9826865422541
44501	1531, 403086353101
964151	172826633439526346998301
3119771	6212414111, 3049724469693931
46061	2081, 87203028281
125731	101, 96911, 5106325320751
204331	170711, 2042244130467251
6571	222731, 152211901
1531	691, 1591242871
2081	275251, 439781
101	491, 1381
170711	12011, 161761, 87422973751
222731	5641, 130039710313651
691	68053211
12011	9065531, 459186991
68053211	14152441, 443279461, 683778634247321
9065531	1301, 1038306890987634192502561

TABLE 3. Table showing prime factors of $\Phi_5(q)$ for some selected primes.

q	$\Phi_5(q)$
44371	60055811, 12908673431
60055811	119701, 1506349456021, 14428681065101
119701	541, 75897500389370461
541	101, 169942181
101	491, 1381
491	191, 603791
1381	811, 1091, 822761
191	1871, 13001
603791	19223261, 44605508992591
811	6781, 12774841
1091	51431, 5514451
822761	16421, 430811, 1177731369781
1871	151, 228729421
13001	1801, 5431, 17981, 32491
19223261	4721, 14951, 4242484381, 91203976571
6781	751411, 1248001
12774841	181, 294288620123733800045450701
51431	241, 1417631, 132128461
5514451	74551, 517954781, 154501318241
16421	3301, 400513020031
430811	6871, 3016901981101
151	104670301
1801	20845133501
5431	174032027589661
17981	3301, 10321, 613680341
32491	34031, 6549693302651
4721	99370619926241
14951	27441881, 364188421
751411	41911, 27248169837781

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