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Implicativity Versus Filtrality, Disjunctivity and Equality Determinants

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IMPLICATIVITY VERSUS FILTRALITY, DISJUNCTIVITY AND EQUALITY DETERMINANTS

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Abstract. The main *general* result of the paper is the fact that a [quasi]variety is (restricted)implicative iff it is [relatively](sub)directly filtral(iff it is [relatively]filtral) iff it is both [relatively]semi-simple and [relatively](sub)directly congruence-distributive, while the class of all its [relatively]simple and one-element members is either a (universal)first-order model class or (both hereditary and)ultra-closed, if(f) it is [relatively]semi-simple and has (R)EDP[R]C, among other things, proved directly without involving any intermediate conceptions like ideality, becoming actually redundant as for the case involved. (This collectively with [4] imply a characterization of discriminator varieties in terms of their most evident and well-known properties.) Likewise, a [quasi]variety is (finitely)restricted disjunctive iff it is [relatively]congruence-distributive, while the class of all its [relatively]finitely-subdirectly-irreducible and one-element members is a universal (first-order)model class, that is, (both ultra-closed and)hereditary. In addition, as a consequence of our properly advanced characterizations of locally-finite restricted implicative/(finitely)disjunctive [quasi]varieties, we prove that a locally finite [quasi]variety is restricted implicative iff it is both (finitely)restricted disjunctive and [relatively]semi-simple. We then apply such characterizations to locally finite varieties of lattice expansions. First of all, we prove that the quasivariety generated by a class of lattice expansions, non-one-element finite subalgebras of which are all simple, is a restricted implicative variety, whenever it is locally finite, its simple/(finitely-)subdirectly-irreducible members being exactly isomorphic copies of non-one-element subalgebras of ultraproducts of members of the class. This covers expansions of both distributive and De Morgan lattices providing, in particular, a *generic* insight into the issue of REDPC for them going back to [7] and [27], respectively. In this connection, we also prove that the variety of De Morgan/Kleene lattices|algebras is not[dual] discriminator. In addition, we prove that the quasivariety generated by a [finite]class of [finite]lattice expansions, non-one-element finite subalgebras of which are all subdirectly-irreducible, is restricted finitely disjunctive, whenever it is a locally finite variety, its (finitely-)subdirectly-irreducibles being (exactly)/[exactly] isomorphic copies of non-one-element subalgebras of ultraproducts of members of the class. Aside from distributive and De Morgan lattices, this is also well-applicable to Stone algebras. And what is more, advancing much further the conception of equality determinant going back to [23] and advanced towards disjunctivity/implicativity in [24]/[26], we provide a *generic constructive* proof of the finite restricted disjunctivity/implicativity of these varieties, equally applicable to certain non-equational quasivarieties studied in [18] and [22]. Finally, as a representative instance of a semi-simple/restricted finitely disjunctive non-implicative variety, we then briefly discuss semilattices/Stone algebras.

1. Introduction

Perhaps, the principal value of *universal* mathematical investigations consists in discovering uniform transparent points behind particular results originally proved *ad hoc*, constructive proofs being definitely preferable. This thesis is a key paradigm of the present study.

On the other hand, the principal problem of *advanced* mathematical investigations is that too complicated formalism and redundant mathematical constructions often hide really transparent points (not saying about gaps) behind the issues under consideration. This equally concerns the study of filtral varieties in [3]. To explain what is *really* wrong with it, we first recall certain closely related issues.

According to [24] and [26], a *(restricted)disjunctive/implicative system for* a class of algebras is a set of(quaternary) equations (normally referred to as a *(restricted)congruence scheme*; cf. [3]) defining disjunction/implication of equalities in algebras of the class, a (quasi)variety being said to be *[finitely|restricted]disjunctive/implicative*, whenever its (relatively)subdirectly-irreducibles have a uniform [finite|restricted]disjunctive/implicative system.[1] It is not especially difficult to show that a [quasi]variety is restricted implicative iff it is [relatively]semi-simple and has *restricted equationally definable principal [relative]congruences* (*REDP[R]C*, for short; cf. [3] for the equational case), the "parameterization" of the "only if" part of this equivalence (more specifically, EDP[R]C for parameterized implicative [quasi]varietis) being far more complicated and remaining a quite non-trivial open problem within this study. Under this equivalence, what was actually announced in [3] is the equivalence of restricted implicativity and subdirect filtrality. Although all was fine with proving REDPC⇔"ideality for subdirect products"⇒"filtrality", the proof of the converse contained a two-fold gap. The thing is that, when dealing with ideality "for subdirect products", *arbitrary subalgebras* of direct products of tuples constituted by *arbitrary members* of the variety are involved (in which case the variety has CEP – this was why all was fine with proving REDPC⇔"ideality for subdirect products" in [3]), while, when dealing with filtrality, *merely subdirect products* of tuples constituted by *merely subdirectly-irreducibles* of the variety are involved. As a matter of fact, it appears that, when restricting our consideration

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[1]In this way, those quasivarieties which are implicative in the sense of [26] are referred to as *restricted* implicative here.

by rather restricted implicativity as such than REDPC in general, involving the conception of ideality becomes redundant, as it is rather expectable intuitively and is definitely shown in this paper on the formal level.

Here, we have refrained from following the paradigm of [3] in proving the equivalence of (sub)direct filtrality and (restricted)implicativity, avoiding involving ideality *at all.* (On the other hand, since the elaboration of [3] admits a quite straightforward "quasi-equational relativization" based mainly upon the inductivity of closure systems of quasivariety-relative congruences resulted from the Compactness Theorem for ultra-closed classes — cf., e.g., [13] – as well as the quite obvious fact that congruences of an algebra of a variety are variety-relative resulted from the fact that varieties are closed under homomorphic images, we equally deal with the quasi-equational framework here well justified by such particular instances as those, which have been studied in [18] and [22] and also discussed here.) Instead, we have found another "bridge" between (restricted)implicativity and [relative](sub)direct filtrality, equally characterizing them but being, as opposed to "[relative]ideality"+"[relative]semi-simplicity", quite valuable in its own right. This is the combination of "[relative](sub)direct congruence-distributivity"+"[relative]semi-simplicity" and either (universal)first-order axiomatizability of the class of all [relatively]simple and one-element algebras of the [quasi]variety or the class' being (both hereditary and)ultra-closed, a [quasi]variety being [relatively]subdirectly congruence-distributive iff it is [relatively]congruence-distributive. In this connection, it is remarkable that a very similar characterization but without involving [relative]semi-simplicity and any kind of filtrality as well as with using [relatively]finitely-subdirectly-irreducibles instead of [relatively]simple members holds for finite restricted disjunctivity. It is such characterizations that have found substantial applications briefly discussed below.

First of all, the described characterization of restricted implicativity collectively with [4] yields a characterization of discriminator varieties in terms of their most evident and well-known properties.

Next, in the *restricted* case of locally-finite (in particular, finitely-generated) quasivarieties, the condition of being ultra-closed can be omitted at all, while the condition of being hereditary can be weakened by replacing it with that of being closed under merely finite subalgebras. This collectively with our characterization of [relatively]finitely-subdirectly-irreducibles of a restricted disjunctive [quasi]variety have enabled us to prove that a locally-finite [quasi]variety is restricted implicative iff it is both (finitely)restricted disjunctive and [relatively]semi-simple. This provides a relationship between restricted implicativity and restricted finite disjunctivity inverse to that given by Remark 2.4 of [26], according to which disjunctive finite restricted systems are definable via implicative ones very much like disjunction is definable via implication in the classical logic $(a \vee b) = (a \supset b) \supset b)$.

Then, most acute advanced definitive *generic* applications concern locally finite varieties of lattice expansions, the congruence-distributivity of which has been well-known due to [17]. First of all, we prove that the quasivariety generated by a class of lattice expansions, non-one-element finite subalgebras of which are all simple, is a restricted implicative variety, whenever it is locally finite, its simple/(finitely-)subdirectly-irreducible members being exactly isomorphic copies of non-one-element subalgebras of ultraproducts of members of the class. This covers expansions of both distributive and De Morgan lattices providing, in particular, a *generic* insight into the issue of REDPC for them going back to [7] and [27], respectively. In addition, we prove that the quasivariety generated by a [finite]class of [finite]lattice expansions, non-one-element finite subalgebras of which are all subdirectly-irreducible, is restricted finitely disjunctive, whenever it is a locally finite variety, its (finitely-)subdirectly-irreducibles being (exactly)/[exactly] isomorphic copies of non-one-element subalgebras of ultraproducts of members of the class. Aside from distributive and De Morgan lattices, this is also well-applicable to Stone algebras. And what is more, advancing much further the conception of equality determinant going back to [23] and advanced towards disjunctivity/implicativity in [24]/[26], we provide a *generic constructive* proof of the finite restricted disjunctivity/implicativity of these varieties, equally applicable to certain non-equational quasivarieties studied in [18] and [22]. (Properly speaking, it is this point that provides a generic uniform *constructive* insight into the issue of REDPC for distributive and De Morgan lattices going back to [7] and [27], respectively, due to providing finite restricted congruence schemes that resemble and have the same number of equations as those found *ad hoc* therein.) Finally, as a representative instance of a semi-simple/restricted finitely disjunctive non-implicative variety, we then briefly discuss semilattices/Stone algebras.

Returning to the issue of uniform constructive proofs of restricted finite disjunctivity/implicativity of quasivarieties under consideration, it goes back to [24] and [26] and is originally based upon the constructive proofs of Lemma 11 and Theorem 12(iii)⇒(i) of [24] and Lemma A.2 of [26], relied, in their turn, upon the conceptions of *equality determinant* and *equational implication* for logical matrices (viz., algebraic systems of the first-order signature resulted from the given algebraic signature by supplementing it with a single unary relation symbol — *truth predicate*) going back to [23], Subsection 7.5 of [24] and Appendix A of [26]. As for matrices, being prime filter lattice expansions, according to Lemma 11 of [24], their algebraic reducts have restricted disjunctive systems found constructively, whenever the matrices as such have equality determinants. This cover many interesting finitely-generated quasi-varieties including expansions of distributive and De Morgan lattices, finitely-valued Lukasiewicz' algebras as well as rather exotic HZ-algebras (cf. [22]) but Sette algebras (cf. [18]), not being congruence-distributive, have no secondary lattice operations (cf. [17]), and so are not covered by the result involved. Likewise, according to Theorem 12(iii)⇒(i) of [24] and Lemma A.2 of [26], algebraic reducts of matrices having equality determinants and equational implications have implicative restricted systems found constructively, whenever they have disjunctive ones. This covers expansions of distributive and Kleene lattices, implicative and bi-lattice expansions of De Morgan lattices, finitely-valued Lukasiewicz' algebras as well as HZ-algebras. However, reductions of Boolean De Morgan algebras (cf. [21]) including De Morgan algebras/lattices are not covered by the result involved, because any expansion of a non-Boolean diamond four-element De Morgan lattice by lattice bounds and/or the complement operation is the algebraic reduct of no matrix with both equality determinant and equational implication. Among other things, implicative systems constructed in this way are sometimes too expansive. It concerns not so much n-valued Łukasiewicz' algebras, equality determinants for which are too expansive (up to $n-1$ elements), as less-many-valued instances. For example, as for Kleene and HZ-algebras, generated by just three-element algebras, the constructed restricted implicative system has 2^{14} equations.

The main purpose of the constructive part of the paper is to eliminate the drawbacks described above. The principal advance results from two novelties. First, we introduce the notion of an *inequality system*, generalizing the usual semi-lattice inequality $(x_0 \lessapprox x_1) \triangleq ((x_0 \wedge x_1) \approx x_0)$, as a set of binary equations holding an analogue of sharing inequalities typical of lattices. Second, we extend (not member-wise) the notions of equality determinant and equational implication to classes of matrices. As a consequence, our advanced approach to disjunctivity well covers Sette algebras. And what is more, as to implicativity, our generic approach covers De Morgan lattices and provides substantial (up to $2^{12/11/11} = (4096/2048/2048)$ times, in case of Kleene/HZ-/Sette algebras, respectively) reduction of the number of equations. As a by-product of our generic elaboration, we come to the conception of equality determinant for a given class of distributive lattice expansions as a uniform equality determinant for all matrices with algebraic reduct belonging to the given class and truth predicate being a prime filter of it. This covers expansions of both distributive and De Morgan lattices as well as finitely-valued Łukasiewicz' algebras. We argue that distributive lattice expansions with finite equality determinant behave very much like algebras with [dual]discriminator do so. In this connection, we also prove that the variety of De Morgan/Kleene lattices|algebras is not[dual] discriminator.

The rest of the paper is as follows. The exposition of the material of the paper is entirely self-contained (of course, modulo very basic issues concerning Set Theory, Lattice Theory, Universal Algebra and Model Theory for both first-order — viz., finitary — and infinitary logics, not specified here explicitly, to be found, e.g., in [1], [13] and [14]). Section 2 is a concise summary of basic issues underlying the paper, most of which have actually become a part of algebraic and logical folklore. Then, Section 3 is a collection of advanced preliminary issues concerning closure sustems with disjunctive basis and disjunctive/implicative systems for algebras. Further, Section 4/5 is entirely devoted to the main general results of the paper concerning the constructive part of the paper described above/ relationship between implicativity and filtrality as well as disjunctivity, respectively, that are then exemplified in Section 6. Finally, Section 7 is a brief summary of principal contributions and open problems of the paper and an outline of further related work.

2. Basic issues

Notations like img, dom, ker, π_i, hom, Con, $\mathbf{I}/\mathbf{S}/\mathbf{H}$ and $\mathbf{P}^{[\mathrm{U/F/SD}]}$ and related notions are supposed to be clear.

2.1. **Set- and lattice-theoretic background.** We follow the standard set-theoretical convention, according to which natural numbers (including 0) are treated as finite ordinals (viz., sets of lesser natural numbers), the ordinal of all them being denoted by ω. The proper class of all ordinals is denoted by ∞.

Likewise, functions are viewed as binary relations, the left/right components of their elements being treated as their arguments/values, respectively. Then, to retain both the conventional prefix writing of functions and the fact that $(f\circ g)(a) = f(g(a))$, we have just preferred to invert the conventional order of relation composition components. In particular, given two binary relations R and Q, we put $R[Q] \triangleq (R \circ Q \circ R^{-1})$, in which case $(\ker R) = R^{-1}[\Delta_{\operatorname{img} R}]$, while $\Delta_K \triangleq \{\langle a, a\rangle | a \in K\}$, whereas K is any class.

In addition, singletons are often identified with their unique elements, unless any confusion is possible.

A function f is said to be *singular*, provided $|\operatorname{img} f| \in 2$, that is, $(\ker f) = (\operatorname{dom} f)^2$.

Let S be a set.

The class (resp., set) of all (sub)sets(of S {including a set A}) [of cardinality $\in K \subseteq \infty$] is denoted by $\wp_{[K]}(\{A,\}S)$. In this way, $\wp$ denotes the *universum*, i.e., the proper class of all sets, subclasses of $\Delta_\wp$ being referred to as *diagonal*. A $T \subseteq S$ is said to be *proper*, if $T \neq S$. Further, given any equivalence relation θ on S, as usual, by ν_θ we denote the function with domain S defined by $\nu_\theta(a) \triangleq [a]_\theta \triangleq \theta[\{a\}]$, for all $a \in S$, in which case $\ker \nu_\theta = \theta$, whereas we set $(T/\theta) \triangleq \nu_\theta[T]$, for every $T \subseteq S$. Next, S-tuples (viz., functions with domain S) are often written in either sequence $\bar{t}$ or vector $\vec{t}$ forms, its s-th component, where $s \in S$, being written as either t_s or t^s in that case. Given two more sets A and B, any relation $R \subseteq (A \times B)$ (in particular, a mapping $R : A \to B$) determines the equally-denoted relation $R \subseteq (A^S \times B^S)$ (resp. mapping $R : A^S \to B^S$) point-wise, that is, $R \triangleq \{\langle \bar{a}, \bar{b}\rangle \in (A^S \times B^S) \mid \forall s \in S : a_s \; R \; b_s\}$. Given a unary operation f on S, put $f^0 \triangleq \Delta_S$ and $f^1 \triangleq f$. Further, set $S^{*[+]} \triangleq \bigcup_{i\in(\omega[\setminus 1])} S^i$. Any binary operation $\diamond$ on S determines the equally-denoted mapping $\diamond : S^+ \to S$ as follows: by induction on the length $l = \operatorname{dom} \bar{a}$ of any $\bar{a} \in S^+$, put:

$$\diamond\bar{a} \triangleq \begin{cases} a_0 & \text{if } l = 1, \\ (\diamond(\bar{a}\upharpoonright(l-1))) \diamond a_{l-1} & \text{otherwise.} \end{cases}$$

As usual, any $\bar{a} \in S^n$, where $n \in \omega$, is identified with the conventional n-tuple $\langle a_0, \ldots, a_{n-1}\rangle$ constructed, by induction on n, in the standard way, via $\varnothing$ and couples (viz., ordered pairs). Likewise, the *concatenation* $\bar{b} * \bar{a}$ *of* any $\bar{b} \in S^*$ *and* $\bar{a}$ is defined, by induction on n, in the standard way. In addition, the *intersection* $\bar{a} \cap K$ *of* $\bar{a}$ *and* any class K is defined, by induction on n, as follows:

$$(\bar{a} \cap K) \triangleq \begin{cases} \bar{a} & \text{if } n = 0, \\ (\bar{a}\upharpoonright(n-1)) \cap K & \text{if } n > 0, a_{n-1} \notin K, \\ \langle (\bar{a}\upharpoonright(n-1)) \cap K, a_{n-1}\rangle & \text{otherwise,} \end{cases}$$

in which case we set $(\bar{a} \setminus K) \triangleq (\bar{a} \cap (\wp \setminus K))$. Finally, given any $n \in (\omega \setminus 1)$ and any $\vec{F} \in \wp(S)^n$, we have $\chi^{\vec{F}} : (\bigcup \operatorname{img} \vec{F}) \to n, a \mapsto \max\{i \in n \mid a \in F_i\}$. In case $n = 2$ and $\vec{F} = \langle S, T\rangle$, where $T \subseteq S$, $\chi^T_S \triangleq \chi^{\vec{F}}$ is the ordinary *characteristic function of* T *in* S.

Let A be a set. A $U \subseteq \wp(A)$ is said to be *upward-directed*, provided, for every $S \in \wp_\omega(U)$, there is some $T \in U$ such that $(\bigcup S) \subseteq T$. A subset of $\wp(A)$ is said to be *inductive*, whenever it is closed under unions of upward-directed subsets. Further, any $X \in T \subseteq \wp(A)$ is said to be *K-meet-irreducible (in/of T)*, where $K \subseteq \infty$, provided it belongs to every $U \in \wp_K(T)$ such that $(A \cap \bigcap U) = X$ (in which case $X \neq A$, whenever $0 \in K$), the set of all them being denoted by $\mathrm{MI}^K(T)$. (Within any context, any mention of K is normally omitted, whenever $K = \infty$. Likewise, "finitely-/pairwise-" means "ω-/$\{2\}$-", respectively.) A *closure system over* A is any $\mathcal{C} \subseteq \wp(A)$ such that, for every $S \subseteq \mathcal{C}$, it holds that $(A \cap \bigcap S) \in \mathcal{C}$, in which case the poset $\langle \mathcal{C}, \subseteq \cap\, \mathcal{C}^2\rangle$ to be identified with $\mathcal{C}$ alone is a complete lattice with meet $A \cap \bigcap$. In that case, any $\mathcal{B} \subseteq \mathcal{C}$ is called a *(closure)basis of* $\mathcal{C}$, provided $\mathcal{C} = \{A \cap \bigcap S | S \subseteq \mathcal{B}\}$. Then, $\mathcal{C}$ is said to be *co-atomic*, provided $\max(\mathcal{C} \setminus \{A\})$ is a basis of it. A *closure operator over* A is any unary operation C on $\wp(A)$ such that, for all $B \subseteq D \in \wp(A)$, it holds that $(C(B) \cup D \cup C(C(D))) \subseteq C(D)$, in which case $\operatorname{img} C$ is a closure system over A, determining C uniquely, because, for every closure basis $\mathcal{B}$ of $\operatorname{img} C$ (in particular, $\operatorname{img} C$ itself) and each $X \subseteq A$, it holds that $C(X) = (A \cap \bigcap\{Y \in \mathcal{B} | X \subseteq Y\})$, and so called *dual to* C and vice versa. For any $X \subseteq A$, $C(X)$ is said to be *generated by* X. Then, elements of $C[\wp_K(A)]$, where $K \subseteq \infty$, are referred to as *K-generated*. (As usual, "principal" means "$\{1\}$-generated".) Further, C is said to be *inductive*, provided, for any upward directed $U \subseteq \wp(A)$, it holds that $C(\bigcup U) \subseteq (\bigcup C[U])$. (Clearly, C is inductive iff $\operatorname{img} C$ is so.)

Remark 2.1. As a consequence of Zorn's Lemma, according to which any inductive non-empty set has a maximal element, given any inductive closure system $\mathcal{C}$, $\mathrm{MI}(\mathcal{C})$ is a closure basis of $\mathcal{C}$, and so is $\mathrm{MI}^K(\mathcal{C}) \supseteq \mathrm{MI}(\mathcal{C})$, where $K \subseteq \infty$. ☐

2.2. **Algebraic and model-theoretic background.** To unify notations, unless otherwise specified, abstract algebras are denoted by capital Fraktur letters (possibly, with indices), their carriers (viz., underlying sets) being denoted by corresponding Italic letters (with same indices, if any). Likewise, unless otherwise specified, we deal with a fixed but arbitrary algebraic (viz., functional) signature Σ constituted by function (viz., operation) symbols of finite arity. Then, algebraic systems (in the sense of [13]) of a first-order signature $\Sigma \cup R$, where R is a relational signature constituted by predicate (viz., relation) symbols of finite arity, are denoted by capital Calligraphic letters (possibly, with indices), their *underlying algebras* (viz., Σ-reducts) being denoted by corresponding Fraktur letters (with same indices, if any).

Given any sub*set* (viz., a subclass not being a *proper* class, i.e., belonging to some class; cf. [14]) S of ∞, put $V_S \triangleq \{x_\beta | \beta \in S\}$ and $(\exists/\forall)_S \triangleq ((\exists/\forall)V_S)$. Then, given any $\alpha \in (\infty \setminus 1)$, by $\mathfrak{Tm}^\alpha_\Sigma$ we denote the absolutely-free Σ-algebra freely-generated by the set V_α, its carrier being denoted by $\mathrm{Tm}^\alpha_\Sigma$. A *$\Sigma$-equation/-identity of rank α* is then any couple of the form $\phi \approx \psi$, where $\phi, \psi \in \mathrm{Tm}^\alpha_\Sigma$, to be identified with the ordered pair $\langle \phi, \psi\rangle$, the set of all them being denoted by $\mathrm{Eq}^\alpha_\Sigma$. A *[first-order/(strict) Horn/positive] Σ-clause of rank α* is then any couple of the form $\Gamma \to \Delta$, where $\Gamma \in \wp_{[\omega/\infty/1]}(\mathrm{Eq}^\alpha_\Sigma)$ and $\Delta \in \wp_{[\omega/2(\setminus 1)/\infty]}(\mathrm{Eq}^\alpha_\Sigma)$ to be identified with Δ, whenever $\Gamma = \varnothing$. (Strict Horn clauses are also called *implications*; cf. [2]. Likewise, first-order implications are also referred to as *quasi-equations/-identities*; cf. [13].) This is treated model-theoretically as the universal sentence $\forall_\alpha((\bigwedge \Gamma) \to (\bigvee \Delta))$. In particular, it is said to be *true/valid/satisfied in* a class K of Σ-algebras, provided it is so in each $\mathfrak{A} \in \mathsf{K}$ *under* every $h \in \hom(\mathfrak{Tm}^\alpha_\Sigma, \mathfrak{A})$ in the sense that $(\Gamma \subseteq (\ker h)) \Rightarrow ((\Delta \cap (\ker h)) \neq \varnothing)$. We also mention the first-order sentences $\Phi_{>1} \triangleq (\exists_2 \neg(x_0 \approx x_1))$ and $\Phi_{\leqslant 2} \triangleq (\forall_3((x_2 \approx x_0) \vee (x_2 \approx x_1)))$ axiomatizing the classes of all non-one-element and no-more-than-two-element Σ-algebras, respectively.

Let $\mathsf{K} \cup \{\mathfrak{A}\}$ be a class of Σ-algebras. Then, K is said to be *locally finite*, provided every finitely-generated member of it is finite.

Given any $\mathcal{H} \subseteq \hom(\mathfrak{A}, \mathsf{K}) \triangleq (\bigcup\{\hom(\mathfrak{A}, \mathfrak{B}) \mid \mathfrak{B} \in \mathsf{K}\})$, put $\ker[\mathcal{H}] \triangleq \{\ker h \mid h \in \mathcal{H}\}$, Then, $\mathcal{H}$ is said to be *discriminating*, provided $(A^2 \cap \bigcap \ker[\mathcal{H}]) \subseteq \Delta_A$, in which case $\hom(\mathfrak{A}, \mathsf{K})$ is so. Next, $\mathfrak{A}$ is said to be *discriminated by* K, provided $\hom(\mathfrak{A}, \mathsf{K})$ is discriminating. Further, elements of $\mathrm{Con}_\mathsf{K}(\mathfrak{A}) \triangleq \{\theta \in \mathrm{Con}(\mathfrak{A}) \mid (\mathfrak{A}/\theta) \in \mathsf{K}\}$ are referred to as *K-(relative)congruences of* $\mathfrak{A}$. (As ν_{Δ_A} is an isomorphism from $\mathfrak{A}$ onto $\mathfrak{A}/\Delta_A$, we then have $(\mathfrak{A} \in \mathsf{K}) \Leftrightarrow (\Delta_A \in \mathrm{Con}_\mathsf{K}(\mathfrak{A}))$, whenever K is closed under $\mathbf{I}$.) In view of the Homomorphism Theorem, we clearly have:

$$\ker[\hom(\mathfrak{A}, \mathsf{K})] = \mathrm{Con}_{\mathbf{IS}\mathsf{K}}(\mathfrak{A}). \tag{2.1}$$

Remark 2.2. For any $\Theta \subseteq \mathrm{Con}_{[\mathsf{K}]}(\mathfrak{A})$, $\theta \triangleq (A^2 \cap \bigcap \Theta) \in \mathrm{Con}(\mathfrak{A})$, while $e^\Theta : (A/\theta) \to \prod_{\vartheta \in \Theta}(A/\vartheta), a \mapsto \langle (\nu_\vartheta \circ \nu_\theta^{-1})(a)\rangle_{\vartheta \in \Theta}$ is an embedding of $(\mathfrak{A}/\theta)$ into $\prod_{\vartheta \in \Theta}(\mathfrak{A}/\vartheta)$ such that, for every $\vartheta \in \Theta$, $(\pi_\vartheta \circ e^\Theta) = (\nu_\vartheta \circ \nu_\theta^{-1}) : (A/\theta) \to (A/\vartheta)$ is a surjection[, in which case $(\mathfrak{A}/\theta) \in \mathbf{IP}^{\mathrm{SD}}\mathsf{K} \subseteq \mathbf{ISP}\mathsf{K}$]. In particular, by the Prime Ideal Theorem, due to which the set of all ultra-filters on a set is a basis of the closure system of all filters on the set,[2] $\mathsf{P}^\mathsf{F}\mathsf{K} \subseteq \mathbf{ISPP}^\mathrm{U}\mathsf{K}$.[3] ☐

In view of Remark 2.2, $\mathrm{Con}_\mathsf{K}(\mathfrak{A})$ is a closure system over A^2, whenever K is closed under $\mathbf{IP}^{\mathrm{SD}}$, in which case the dual closure operator (of relative congruence generation) is denoted by $\mathrm{Cg}^\mathfrak{A}_\mathsf{K}$. (The subscript K is allowed to be omitted, whenever $\mathsf{K} \ni \mathfrak{A}$ is a variety [in particular, the one of all Σ-algebras], in which case $\mathrm{Con}_\mathsf{K}(\mathfrak{A}) = \mathrm{Con}(\mathfrak{A})$.) In view of (2.1), we then have:

Proposition 2.3. *Suppose* K *is closed under* $\mathbf{ISP}$. *Then, any Σ-implication $\Gamma \to \Phi$ of rank $\alpha \in (\infty \setminus 1)$ is true in* K *iff* $\Phi \in \mathrm{Cg}^{\mathfrak{Tm}^\alpha_\Sigma}_\mathsf{K}(\Gamma)$.

Remark 2.4 (cf. Remark 1.2 of [26]). If $\mathfrak{A} \in \mathbf{ISP}\mathsf{K}$, in which case there are some set I, some $\overline{\mathfrak{B}} \in \mathsf{K}^I$ and some embedding e of $\mathfrak{A}$ into $\prod_{i \in I} \mathfrak{B}_i$, then $\{\pi_i \circ e \mid i \in I\} \subseteq \hom(\mathfrak{A}, \mathsf{K})$ is discriminating. Conversely, if $\mathfrak{A}$ is discriminated by K, then, by (2.1), we have $(A^2 \cap \bigcap \Theta) = \Delta_A$, where $\Theta \triangleq \mathrm{Con}_{\mathbf{IS}\mathsf{K}}(\mathfrak{A})$, in which case, by Remark 2.2, we get $\mathfrak{A} \in \mathbf{IP}^{\mathrm{SD}}\mathbf{IS}\mathsf{K} \subseteq \mathbf{ISP}\mathsf{K}$ (and what is more, if either $\mathfrak{A}$ is finite, in which case Θ is so, or both $\mathfrak{A}$ is finitely-generated and both K and all members of it are finite,

[2] As opposed to ultra-filters, filters are naturally allowed to contain the empty set, while products of empty tuples are equally allowed, contrary to the rather odd conventions accepted, e.g., in [13].

[3] This well-known fact is used tacitly throughout the paper.

in which case $\hom(\mathfrak{A},\mathsf{K})$ is finite, and so is Θ, in view of (2.1), then merely [sub]direct products of finite tuples are needed, and so, in particular, $\mathbf{ISP}\mathsf{K}$ is locally finite, whenever both K and all members of it are finite[4]). Thus, $\mathbf{ISP}\mathsf{K}$ is the class of all Σ-algebras discriminated by K. □

Lemma 2.5 (cf. [2]([13]) for the (positive)strict Horn case)**.** $[(\mathbf{H})]\mathbf{IS}[\mathbf{P}]\mathsf{K}$ *is axiomatized by the class of all [(positive)strict Horn]Σ-clauses true in* K*. In particular,* K *is universally [(equationally) implicationally] axiomatizable iff it is closed* $[(\mathbf{H})]\mathbf{IS}[\mathbf{P}]$.

Proof. Clearly, any [(positive)strict Horn] Σ-clause being true in K is also true in $[(\mathbf{H})]\mathbf{IS}[\mathbf{P}]\mathsf{K}$. Conversely, assume $\mathfrak{A}\notin[(\mathbf{H})]\mathbf{IS}[\mathbf{P}]\mathsf{K}$. Then, $\alpha\triangleq|A|\in(\infty\setminus 1)$. Take any bijection $h:V_\alpha\to A$ to be extended to the equally denoted surjective homomorphism from $\mathfrak{T}\triangleq\mathfrak{Tm}^\alpha_\Sigma$ onto $\mathfrak{A}$. Put $\Gamma\triangleq(\ker h)$ and $\Delta\triangleq(\mathrm{Eq}^\alpha_\Sigma\setminus\Gamma)$. Let us show, by contradiction, that the Σ-clause $\Gamma\to\Delta$ [resp., for some $\Phi\in\Delta$, the Σ-implication(-equation) $\Gamma(\varnothing)\to\Phi$] of rank α is true in K. For suppose there are some $\mathfrak{B}\in\mathsf{K}$ and some $g\in\hom(\mathfrak{T},\mathfrak{B})$ such that $\Gamma\subseteq(\ker g)$, while $(\Delta\cap(\ker g))=\varnothing$, in which case $(\ker g)=(\ker h)$, and so, by the Homomorphism Theorem, $g\circ h^{-1}$ is an embedding of $\mathfrak{A}$ into $\mathfrak{B}$, in which case $\mathfrak{A}\in\mathbf{IS}\mathsf{K}$.[Respectively, suppose, for every $\Phi\in\Delta$, $\Gamma\to\Phi$ is not true in K. Consider any $\vec{a}\in(A^2\setminus\Delta_A)$. Then, there is some $\vec{\varphi}\in\Delta$ such that $h(\vec{\varphi})=\vec{a}$, in which case $\Gamma\to\vec{\varphi}$ is not true in K, and so there are some $\mathfrak{B}\in\mathsf{K}$ and some $g\in\hom(\mathfrak{T},\mathfrak{B})$ such that $\Gamma\subseteq(\ker g)$, while $\vec{\varphi}\notin(\ker g)$. Therefore, by the Homomorphism Theorem, $e\triangleq(g\circ h^{-1})\in\hom(\mathfrak{A},\mathfrak{B})$, in which case $g=(e\circ h)$, and so $e(a_0)\neq e(a_1)$. Thus, $\mathfrak{A}$ is discriminated by K. Hence, by Remark 2.4, $\mathfrak{A}\in\mathbf{ISP}\mathsf{K}$.(Likewise, suppose every $\Phi\in\Delta$ is not true in $\mathsf{K}\subseteq\mathbf{IP}\mathsf{K}\subseteq\mathbf{ISP}\mathsf{K}$, the latter class being closed under $\mathbf{ISP}$. Then, by Proposition 2.3, $\theta\triangleq\mathrm{Cg}^{\mathfrak{T}}_{\mathbf{ISP}\mathsf{K}}(\varnothing)\subseteq(\ker h)$. Therefore, by the Homomorphism Theorem, $h\circ\nu_\theta^{-1}$ is a surjective homomorphism from $(\mathfrak{T}/\theta)\in\mathbf{ISP}\mathsf{K}$ onto $\mathfrak{A}$. Hence, $\mathfrak{A}\in\mathbf{HISP}\mathsf{K}$.)] This contradiction does imply that $\Gamma\to\Delta$ [resp., for some $\Phi\in\Delta$, $\Gamma(\varnothing)\to\Phi$] is true in K. On the other hand, $\Gamma\to\Delta[(\varnothing)\to\Phi]$ is not true in $\mathfrak{A}$ under h, as required. □

Here, we need solely the following clause particular case of the generic Compactness Theorem (cf., e.g., [13]):

Theorem 2.6 (Clause Compactness Theorem)**.** *Let* $\Gamma\to\Delta$ *be a* Σ*-clause true in* $\mathbf{P}^{\mathrm{U}}\mathsf{K}$*. Then, there are some* $\Xi\in\wp_\omega(\Gamma)$ *and some* $\Theta\in\wp_\omega(\Delta)$ *such that the first-order* Σ*-clause* $\Xi\to\Theta$ *is true in* K*, and so in* $\mathbf{P}^{\mathrm{U}}\mathsf{K}$.

Combining Lemma 2.5 and Theorem 2.6, we immediately get:

Corollary 2.7 (cf. [13])**.** $\mathbf{IS}[\mathbf{P}]\mathbf{P}^{\mathrm{U}}\mathsf{K}$ *is axiomatized by the set of all first-order[strict Horn]* Σ*-clauses of rank* ω *true in* K*. In particular,* K *is universally first-order-[strictly-Horn-]axiomatizable iff it is closed under* $\mathbf{IS}[\mathbf{P}]\mathbf{P}^{\mathrm{U}}$.

A/An *[quasi-]variety/ implicational class* (viz., a *prevariety* in terms of [28]) is any class of Σ-algebras, axiomatized by a class of Σ-[quasi-]identities/-implications or, equivalently (cf. Lemma 2.5 and Corollary 2.7), closed under $\mathbf{H}$, $\mathbf{S}$ and $\mathbf{P}$ [resp., $\mathbf{I}$, $\mathbf{S}$, $\mathbf{P}$ and $\mathbf{P}^{\mathrm{U}}$]/ resp., $\mathbf{I}$, $\mathbf{S}$ and $\mathbf{P}$.

Lemma 2.8 (Relative Correspondence Theorem)**.** *Let* $\mathfrak{B}$ *be a* Σ*-algebra and* $h\in\hom(\mathfrak{A},\mathfrak{B})$*.(Suppose* $\mathrm{img}\,h=B$*.)[Assume* K *is closed under* $\mathbf{IS}$ *(resp.,* $\mathbf{I}$*)]. Then,* $h^{-1}[\vartheta]\in(\mathrm{Con}_{[\mathsf{K}]}(\mathfrak{A})\cap\wp(\ker h,A^2))$*, for all* $\vartheta\in\mathrm{Con}_{[\mathsf{K}]}(\mathfrak{B})$*(, whereas* $h[h^{-1}[\vartheta]]=\vartheta$*, while* $h[\theta]\in\mathrm{Con}_{[\mathsf{K}]}(\mathfrak{B})$*, for all* $\theta\in(\mathrm{Con}_{[\mathsf{K}]}(\mathfrak{A})\cap\wp(\ker h,A^2))$*, whereas* $h^{-1}[h[\theta]]=\theta$*, in which case the mappings* $\theta\mapsto h[\theta]$*, preserving unions, and* $\vartheta\mapsto h^{-1}[\vartheta]$*, preserving both unions and intersections, are inverse to one another isomorphisms between the posets* $\mathrm{Con}_{[\mathsf{K}]}(\mathfrak{A})\cap\wp(\ker h,A^2)$ *and* $\mathrm{Con}_{[\mathsf{K}]}(\mathfrak{B})$ *ordered by inclusion).*

Proof. The "[]"-option-free (viz., "absolute") case is well-known.

[Finally, consider any $\vartheta\in\mathrm{Con}(\mathfrak{B})$. Then, $g\triangleq(\nu_\vartheta\circ h)\in\hom(\mathfrak{A},\mathfrak{B}/\vartheta)$(is surjective), while $\ker g=\theta\triangleq h^{-1}[\vartheta]\in\mathrm{Con}(\mathfrak{A})$. Hence, by the Homomorphism Theorem, $g\circ\nu_\theta^{-1}$ is an embedding of (resp., isomorphism from) $\mathfrak{A}/\theta$ into (resp., onto) $\mathfrak{B}/\vartheta$, so $(\theta\in\mathrm{Con}_{\mathsf{K}}(\mathfrak{A}))\Leftarrow(\Leftrightarrow)(\vartheta\in\mathrm{Con}_{\mathsf{K}}(\mathfrak{B}))$, as required.] □

Further, $\mathfrak{A}$ is said to be *[*K*-(relatively)]simple/*K*-subdirectly-irreducible*, where $K\subseteq\infty$, if $\mathrm{Con}_{[\mathsf{K}]}(\mathfrak{A})\subseteq\{\Delta_A,A^2\}\neq\{A^2\}$/resp., $[(\Delta_A\in\mathrm{Con}_{\mathsf{K}}(\mathfrak{A}))\Rightarrow](\Delta_A\in\mathrm{MI}^K(\mathrm{Con}_{[\mathsf{K}]}(\mathfrak{A})))$. Next[, in case K is closed under $\mathbf{IP}^{\mathrm{SD}}$], $\mathfrak{A}$ is said to be *[*K*-(relatively)]congruence-distributive/-modular*, provided the lattice $\mathrm{Con}_{[\mathsf{K}]}(\mathfrak{A})$ is distributive/modular. Furthermore, $\mathfrak{A}$ is said to be *congruence-permutable*, provided any $\theta,\vartheta\in\mathrm{Con}(\mathfrak{A})$ are *permutable* in the sense that $(\theta\circ\vartheta)\subseteq(\vartheta\circ\theta)$, in which case $(\theta\vee\vartheta)=(\theta\circ\vartheta)$, and so $\mathfrak{A}$ is congruence-modular. Finally, $\mathfrak{A}$ is said to be *arithmetical*, whenever it is both congruence-distributive and congruence-permutable.

The following generic observation is equally applicable to both semilattices and quasi-Sette algebras (cf. [18]), as examples with non-empty signatures (cf. Subsections 6.6 and 6.7, respectively):

Corollary 2.9. *Suppose* $A=2$ *and* $\hbar:2^2\to 2^2,\langle i,j\rangle\mapsto\langle i,\min(i,j)\rangle$ *is an endomomorphism of* $\mathfrak{A}^2$ *(in particular,* $\Sigma=\varnothing$*). Then,* $\mathfrak{A}^2$ *is not congruence-modular, and so neither congruence-permutable nor congruence-distributive.*

Proof. Clearly, $g_k\triangleq(\pi_k\restriction A^2)$, where $k\in 2$, is a surjective homomorphism from $\mathfrak{A}^2$ onto $\mathfrak{A}$, in which case, by Lemma 2.8, $\vartheta\triangleq(\ker\hbar)\in\mathrm{Con}(\mathfrak{A}^2)$, while $\theta_k\triangleq(\ker g_k)\in\max(\mathrm{Con}(\mathfrak{A}^2)\setminus\{(A^2)^2\})$, for $\mathfrak{A}$, being two-element, is simple. Moreover, $\theta_0\supsetneq\vartheta\nsubseteq\theta_1$, in which case $(\vartheta\vee\theta_1)=(A^2)^2$, while $(\theta_0\cap\theta_1)=\Delta_{A^2}$. Therefore, $(\vartheta\vee(\theta_0\cap\theta_1))=\vartheta\neq\theta_0=(\theta_0\cap(\vartheta\vee\theta_1))$, as required. □

[4]This well-known fact is used tacitly throughout the paper.

Next, K is said to be *congruence-permutable/arithmetical*, whenever every member of it is so. Likewise[, in case K is closed under $\mathbf{IP}^{\mathrm{SD}}$], K is said to be *[relatively]congruence-distributive/-modular*, whenever every member of it is [K-relatively]congruence-distributive/-modular. Next, the class of all finite/([K-]simple/K-subdirectly-irreducible, where $K \subseteq \infty$) members of K is denoted, respectively, by $\mathsf{K}_{<\omega}/(\mathrm{Si}\,/\,\mathrm{SI}^{K})_{[\mathsf{K}]}(\mathsf{K})$. Further, the [quasi/pre]variety $\mathbf{HSP}\mathsf{K}$ [$\mathbf{ISPP}^{\mathrm{U}}\mathsf{K}/\mathbf{ISP}\mathsf{K}$, respectively] generated by K (i.e., the least one including K; cf. Lemma 2.5 and Corollary 2.7) is denoted by $[\mathbf{Q}/\mathbf{P}]\mathbf{V}(\mathsf{K})$, respectively, and said to be *finitely-generated*, whenever both K and all members of it are finite, in which case $\mathbf{QV}(\mathsf{K}) = \mathbf{PV}(\mathsf{K})$ is locally-finite. (Note that $\mathbf{V}(\mathsf{K}) = \mathbf{V}((\mathbf{Q}/\mathbf{P})\mathbf{V}(\mathsf{K})) = \mathbf{H}(\mathbf{Q}/\mathbf{P})\mathbf{V}(\mathsf{K})$, in which case $\mathbf{V}(\mathsf{K})$ is locally finite iff $(\mathbf{Q}/\mathbf{P})\mathbf{V}(\mathsf{K})$ is so.) Finally, the class of all non-one-element subalgebras/homomorphic images of members of K is denoted by $(\mathbf{S}/\mathbf{H})_{>1}\mathsf{K}$, respectively. In view of Lemma 2.8, we first have:

$$\mathrm{SI}_{[\mathbf{PV}(\mathsf{K})]}(\mathbf{PV}(\mathsf{K})) \subseteq \mathbf{IS}_{>1}\mathsf{K}. \tag{2.2}$$

As an immediate consequence of Lemma 2.8, we also have:

Corollary 2.10. *Let $K \subseteq \infty$. Suppose K is closed under $\mathbf{I}$. Then, $(\mathrm{Si}\,/\,\mathrm{SI}^{K})_{[\mathsf{K}]}(\mathsf{K})$ is so.*

Corollary 2.11. *Let $\mathfrak{B}$ be a Σ-algebra and $h \in \hom(\mathfrak{A}, \mathfrak{B})$.(Suppose $\mathrm{img}\, h = B$.)[Assume K is closed under $\mathbf{ISP}$ (resp., $\mathbf{IP}^{\mathrm{SD}}$).] Then, for all $X \subseteq A^2$, it holds that $\mathrm{Cg}^{\mathfrak{A}}_{[\mathsf{K}]}(X \cup (\ker h)) \subseteq (=) h^{-1}[\mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{K}]}(h[X])]$.*

Proof. By Lemma 2.8, we then have

$$h^{-1}[\mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{K}]}(h[X])] = h^{-1}[B^2 \cap \bigcap\{\vartheta \in \mathrm{Con}_{[\mathsf{K}]}(\mathfrak{B}) | h[X] \subseteq \vartheta\}] =$$
$$(A^2 \cap \bigcap\{h^{-1}[\vartheta] | \vartheta \in \mathrm{Con}_{[\mathsf{K}]}(\mathfrak{B}), X \subseteq h^{-1}[\vartheta]\}) \supseteq (=)$$
$$(A^2 \cap \bigcap\{\theta \in \mathrm{Con}_{[\mathsf{K}]}(\mathfrak{A}) | (X \cup (\ker h)) \subseteq \theta\}) = \mathrm{Cg}^{\mathfrak{A}}_{[\mathsf{K}]}(X \cup (\ker h)),$$

as required. □

Corollary 2.12. *Suppose K is a prevariety. Then, the following are equivalent:*

(i) *K is a quasivariety;*
(ii) *for every Σ-algebra $\mathfrak{A}$, the closure system $\mathrm{Con}_{\mathsf{K}}(\mathfrak{A})$ is inductive;*
(iii) *for every $\mathfrak{A} \in \mathsf{K}$, the closure system $\mathrm{Con}_{\mathsf{K}}(\mathfrak{A})$ is inductive.*

Proof. First, (ii)⇒(i) is by Proposition 2.3 and the "[]"-optional case of Lemma 2.5. Conversely, assume (i) holds. Consider any Σ-algebra $\mathfrak{A}$. Let $\alpha \triangleq |A| \in (\infty \setminus 1)$. Take any bijection $h : V_\alpha \to A$ to be extended to the equally-denoted surjective homomorphism from $\mathfrak{T} \triangleq \mathfrak{Tm}^{\alpha}_{\Sigma}$ onto $\mathfrak{A}$. Then, by Proposition 2.3 and Theorem 2.6, $\mathrm{Cg}^{\mathfrak{T}}_{\mathsf{K}}$ is inductive, in which case $\mathrm{Con}_{\mathsf{K}}(\mathfrak{T})$ is inductive, and so is $\mathrm{Con}_{\mathsf{K}}(\mathfrak{T}) \cap \wp(\ker h, T^2)$. In this way, Lemma 2.8 yields (ii).

Next, (iii) is a particular case of (ii). Conversely, assume (iii) holds. Consider any Σ-algebra $\mathfrak{A}$. Then, $\theta \triangleq \mathrm{Cg}^{\mathfrak{A}}_{\mathsf{K}}(\varnothing) \in \mathrm{Con}_{\mathsf{K}}(\mathfrak{A})$, in which case $h \triangleq \nu_\theta$ is a surjective homomorphism from $\mathfrak{A}$ onto $\mathfrak{B} \triangleq (\mathfrak{A}/\theta) \in \mathsf{K}$, while $(\ker h) = \theta$, whereas $\mathrm{Con}_{\mathsf{K}}(\mathfrak{A}) \subseteq \wp(\theta, A^2)$, and so (iii) and Lemma 2.8 imply (ii), as required. □

By Remarks 2.1, 2.2, Lemma 2.8 and Corollary 2.12, we also have:

Theorem 2.13 (Relative Subdirect Product Representation Theorem)**.** *Let $K \subseteq \infty$. Suppose K is a [quasi]variety. Then, $\mathsf{K} = \mathbf{IP}^{\mathrm{SD}}\,\mathrm{SI}^{K}_{[\mathsf{K}]}(\mathsf{K})$.*

A quasivariety Q of similar algebras is said to be *[relatively]semi-simple*, provided $\mathsf{Q} \subseteq \mathbf{IP}^{\mathrm{SD}}\,\mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q})$ or, equivalently (in view of Corollary 2.10 and Theorem 2.13 with $K = \infty$), $\mathrm{SI}_{\mathsf{Q}}(\mathsf{Q}) \subseteq \mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q}) (\subseteq \mathrm{Si}_{\mathsf{Q}}(\mathsf{Q}))$,[5] in which case, by Lemma 2.8, we have $\mathrm{MI}(\mathrm{Con}_{\mathsf{Q}}(\mathfrak{A})) \subseteq \max(\mathrm{Con}_{\mathsf{Q}}(\mathfrak{A}) \setminus \{A^2\})$, for each $\mathfrak{A} \in \mathsf{Q}$, and so, by Remark 2.1 and Corollary 2.12, $\mathrm{Con}_{\mathsf{Q}}(\mathfrak{A})$ is co-atomic[, while the converse is by Lemma 2.8 and the fact that any basis of any closure system $\mathcal{C}$ includes $\mathrm{MI}(\mathcal{C})$].

Let I be a set and $\overline{\mathfrak{A}}$ an I-tuple of Σ-algebras. Given any $\bar{a}, \bar{b} \in \prod_{i \in I} A_i$, as usual, $E(\bar{a}, \bar{b})$ denotes their *equalizer* $\{i \in I | a_i = b_i\}$. Then, given any $\mathfrak{B} \in \mathbf{S}(\prod_{i \in I} \mathfrak{A}_i)$, a $\theta \in \mathrm{Con}(\mathfrak{B})$ is said to be *[principally /ultra-]filtral*, provided there is a [principal /ultra-]filter $\mathcal{F}$ on I such that $\theta = \theta^B_{\mathcal{F}} \triangleq \{\langle \bar{a}, \bar{b} \rangle \in B^2 | E(\bar{a}, \bar{b}) \in \mathcal{F}\}$.

Lemma 2.14 (Relative Ultra-filter Lemma)**.** *Let I be a set, $\overline{\mathfrak{A}}$ an I-tuple of Σ-algebras, $\mathfrak{B} \in \mathbf{S}(\prod_{i \in I} \mathfrak{A}_i)$[, Q a quasivariety of Σ-algebras], $\theta \in \mathrm{Con}_{[\mathsf{Q}]}(\mathfrak{B})$ and $K \subseteq \infty$. Suppose $\{0, 2\} \subseteq K$ and $\mathfrak{B}/\theta$ is [Q-relatively]both K-subdirectly irreducible and congruence-distributive. Then, there is some ultrafiltral $\vartheta \in (\mathrm{Con}(\mathfrak{B}) \cap \wp(\theta))$.*

Proof. [In view of Lemma 2.8 with $h = \Delta_B$, any filtral congruence of $\mathfrak{B}$ is Q-relative.]Using Lemma 2.8 and[relativizing (via involving Q-relative congruences of $\mathfrak{B}$ instead of arbitrary ones)] the proof of Theorem 2.6 of [16], we get a needed proof, the condition $0 \in K$ implying the fact that $I \neq \varnothing$. □

By the Homomorphism Theorem and Lemmas 2.8 and 2.14,[since finite finitely-subdirectly-irreducible algebras are subdirectly-irreducible,] we first have:

Corollary 2.15. *Let K be a[finite] class of[finite] similar algebras and $K \subseteq \infty$. Suppose $\mathsf{V} \triangleq \mathbf{V}(\mathsf{K})$ is congruence-distributive and $\{0, 2\} \subseteq K$. Then, $\mathrm{SI}^{K}(\mathsf{V})[= \mathrm{SI}(\mathsf{V})] \subseteq \mathbf{H}_{>1}\mathbf{S}_{>1}\mathbf{P}^{\mathrm{U}}\mathsf{K}[= \mathbf{H}_{>1}\mathbf{S}_{>1}\mathsf{K}]$.*

[5] It is in this way that we come to a well-justified notion of *absolutely* semi-simple *quasi*variety. In this connection, just recall that the right meaning of semi-simplicity is (subdirect) representability by simple algebras.

And what is more, we also have:

Corollary 2.16. *Let* K *be a[finite] class of[finite] similar algebras and* $K \subseteq \infty$. *Suppose* $\mathsf{Q} \triangleq \mathbf{QV}(\mathsf{K})[= \mathbf{PV}(\mathsf{K})]$ *is relatively congruence-distributive and* $\{0,2\} \subseteq K$. *Then,* $\mathrm{SI}^K_{\mathsf{Q}}(\mathsf{Q})[= \mathrm{SI}_{\mathsf{Q}}(\mathsf{Q})] \subseteq \mathbf{IS}_{>1}\mathbf{P}^{\mathrm{U}}\mathsf{K}[= \mathbf{IS}_{>1}\mathsf{K}]$.

Proof. By Lemmas 2.8, 2.14 and Corollary 2.7, we have $\mathrm{SI}^K_{\mathsf{Q}}(\mathsf{Q}) = \mathrm{SI}^K_{\mathsf{Q}}(\mathbf{ISPP}^{\mathrm{U}}\mathsf{K}) \subseteq \mathbf{IS}_{>1}\mathbf{P}^{\mathrm{U}}\mathbf{P}^{\mathrm{U}}\mathsf{K} \subseteq \mathbf{IS}_{>1}\mathbf{P}^{\mathrm{U}}\mathbf{ISP}^{\mathrm{U}}\mathsf{K} \subseteq \mathbf{IS}_{>1}\mathbf{ISP}^{\mathrm{U}}\mathsf{K} \subseteq \mathbf{IS}_{>1}\mathbf{P}^{\mathrm{U}}\mathsf{K}[= \mathbf{IS}_{>1}\mathsf{K}$, in which case the fact that finite Q-relatively finitely-subdirectly-irreducible algebras are Q-relatively subdirectly-irreducible completes the argument]. ☐

Corollary 2.17. *Let* K *be a[finite] class of[finite] similar algebras. Suppose* $\mathsf{Q} \triangleq \mathbf{QV}(\mathsf{K})[= \mathbf{PV}(\mathsf{K})]$ *is congruence-distributive and every member of* $\mathbf{S}_{>1}\mathbf{P}^{\mathrm{U}}\mathsf{K}$ *[resp.,* $\mathbf{S}_{>1}\mathsf{K}$*] is simple. Then,* Q *is a semi-simple variety.*

Proof. In that case, by Lemma 2.8, $\mathsf{V} \triangleq \mathbf{V}(\mathsf{K}) = \mathbf{H}\mathsf{Q}$ is congruence-distributive. Then, by Lemma 2.8 and Corollaries 2.10 and 2.15 with $K = \infty$, we have $\mathrm{SI}(\mathsf{V}) \subseteq \mathbf{IS}_{>1}\mathbf{P}^{\mathrm{U}}\mathsf{K}$ $[\mathrm{SI}(\mathsf{V}) \subseteq \mathbf{IS}_{>1}\mathsf{K}] \subseteq \mathbf{I}\,\mathrm{Si}(\mathsf{Q}) \subseteq \mathrm{Si}(\mathsf{Q})$. Hence, by Theorem 2.13 with $K = \infty$, we eventually get $\mathsf{V} \subseteq \mathsf{Q} \subseteq \mathsf{V}$, as required. ☐

In this connection, recall also the following quite useful auxiliary observation:

Lemma 2.18 (Lemma 2.1 of [26]). *Let* Q *be a locally-finite quasivariety and* $\mathsf{K} \subseteq \mathsf{Q}$. *Then,* $(\mathbf{SP}^{\mathrm{U}}\mathsf{K})_{<\omega} \subseteq \mathbf{IS}\mathsf{K}$.

Corollary 2.19. *Let* Q *be a locally-finite quasivariety,* $\mathsf{K} \subseteq \mathsf{Q}$ *and* $\mathsf{S} \subseteq \mathsf{Q}$. *Suppose* S *is closed under* $\mathbf{I}$, $(\mathbf{S}_{>1}\mathsf{K})_{<\omega} \subseteq \mathsf{S}$ *and, for every* $\mathfrak{A} \in (\mathsf{Q} \setminus \mathbf{V}(\varnothing))$, *it holds that* $((\mathbf{S}_{>1}\mathfrak{A})_{<\omega} \subseteq \mathsf{S}) \Rightarrow (\mathfrak{A} \in \mathsf{S})$. *Then,* $\mathbf{S}_{>1}\mathbf{P}^{\mathrm{U}}\mathsf{K} \subseteq \mathsf{S}$.

Proof. Consider any $\mathfrak{A} \in \mathbf{S}_{>1}\mathbf{P}^{\mathrm{U}}\mathsf{K}$, in which case it is not one-element, and any finite non-one-element subalgebra $\mathfrak{B}$ of $\mathfrak{A}$. Then, by Lemma 2.18, we have $\mathfrak{B} \in (\mathbf{S}_{>1}\mathbf{P}^{\mathrm{U}}\mathsf{K})_{<\omega} \subseteq \mathbf{I}(\mathbf{S}_{>1}\mathsf{K})_{<\omega} \subseteq \mathbf{I}\mathsf{S} \subseteq \mathsf{S}$, and so $\mathfrak{A} \in \mathsf{S}$, as required. ☐

A [quasi]variety Q of Σ-algebras is said to be *[relatively]{(sub)directly }filtral*, provided, for every set I, all [Q-]congruences of each subalgebra of the direct product of any $\overline{\mathfrak{A}} \in \mathrm{SI}_{[\mathsf{Q}]}(\mathsf{Q})^I${, being a (sub)direct product of the I-tuple involved,} are filtral, in which case, by Corollary 2.10, Q is [relatively]semi-simple, because $\mathfrak{A}$ is isomorphic to the direct power $\mathfrak{A}^1$, while $\{1\}$ and $\wp(1)$ are the only filters on 1. (It is subdirect filtrality that corresponds to the filtrality in the comprehension of [3], while the direct filtrality is beyond its scopes *at all.*)

Likewise, a [quasi]variety Q of Σ-algebras is said to be *[relatively](sub)directly congruence-distributive/-modular*, provided, for every set I, each (sub)direct product of any $\overline{\mathfrak{A}} \in \mathrm{SI}_{[\mathsf{Q}]}(\mathsf{Q})^I$ is [Q-]congruence-distributive/-modular. (In view of Lemma 2.8 and Theorem 2.13, a [quasi]variety is [relatively]congruence-distributive/-modular iff it is subdirectly so.)

Lemma 2.20 (Subalgebra Lemma; cf. [9]). *Let* $\mathfrak{B}$ *a subalgebra of* $\mathfrak{A}$. *Then,* $h \triangleq \{\langle f, b\rangle \in (A^\omega \times B) | \exists F \in \wp_\omega(\omega) : f[\omega \setminus F] \subseteq \{b\}\}$ *is a function forming a subalgebra of* $\mathfrak{A}^\omega \times \mathfrak{B}$, *in which case* $\mathrm{dom}\, h$ *forms a subalgebra of* $\mathfrak{A}^\omega$, *while* $h \in \hom(\mathfrak{C}, \mathfrak{A})$, *where* $\mathfrak{C} \triangleq (\mathfrak{A}^\omega {\restriction} (\mathrm{dom}\, h))$. *Moreover,* $\pi_i[C] = A$, *for all* $i \in \omega$, *whereas, for all* $b \in B$, $\langle \omega \times \{b\}, b\rangle \in h$, *and so* $h[C] = B$.

Corollary 2.21. *Let* Q *be a [relatively]subdirectly filtral [quasi]variety. Then,* $\mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q})$ *is closed under* $\mathbf{S}_{>1}$.

Proof. Consider any $\mathfrak{A} \in \mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q}) \subseteq \mathrm{SI}_{[\mathsf{Q}]}(\mathsf{Q})$, any $\mathfrak{B} \in \mathbf{S}_{>1}\mathfrak{A}$ and an arbitrary $\theta \in (\mathrm{Con}_{[\mathsf{Q}]}(\mathfrak{B}) \setminus \{\Delta_B\})$. Let $\mathfrak{C}$ and h be as in Lemma 2.20. Then, by Lemma 2.8, $h^{-1}[\theta] \in \mathrm{Con}_{[\mathsf{Q}]}(\mathfrak{C})$ and $h[h^{-1}[\theta]] = \theta$, in which case, by the [relative]subdirect filtrality of Q, there is some filter $\mathcal{F}$ on ω such that $\theta^{\mathcal{C}}_{\mathcal{F}} = h^{-1}[\theta]$. Take any $\langle a, b\rangle \in (\theta \setminus \Delta_B) \neq \varnothing$. Then, $\langle \omega \times \{a\}, \omega \times \{b\}\rangle \in h^{-1}[\theta]$. On the other hand, $E(\omega \times \{a\}, \omega \times \{b\}) = \varnothing$, for $a \neq b$, in which case $\mathcal{F} = \wp(\omega)$, and so $h^{-1}[\theta] = C^2$. In this way, we eventually get $\theta = h[C^2] = B^2$, as required. ☐

A *(congruence-)permutation term for* K [12] is any $\varpi \in \mathrm{Tm}^3_\Sigma$ such that the identities of the form $x_{2\cdot(1-k)} \approx (\varpi[x_{2\cdot k}/x_1])$, where $k \in 2$, are true in K, in which case K is congruence-permutable.

A *(Pixley)majority/minority term for* K [17] is any $\mu \in \mathrm{Tm}^3_\Sigma$ such that the following identities are true in K:

$$x_{l(i)} \approx (\mu[x_i/x_1, x_j/x_0]_{j \in (3 \setminus \{i\})}), \tag{2.3}$$

where $i \in 3$ and $l(i) \triangleq 0/(1 - \min(2 - i, i))$, in which case $\varpi[x_1/\mu]$, where ϖ is a permutation term for K, is, conversely, a minority/majority term for K, while K is congruence-distributive/arithmetical, because any minority term for K is a permutation one.

Let $\Sigma^+_{[\frac{1}{2}]} \triangleq \{\wedge, \vee\}[\{\wedge\}]$ be the ordinary [semi]lattice signature, where $\wedge$ and $\vee$ are binary. Given any $\Sigma \supseteq \Sigma^+_{\frac{1}{2}}$, $\phi \lessapprox \psi$ is used as an abbreviation for $(\phi \wedge \psi) \approx \phi$, where ϕ and ψ are Σ-terms. Then, given any Σ-algebra $\mathfrak{A}$ such that $\mathfrak{A}{\restriction}\Sigma^+_{\frac{1}{2}}$ is a semilattice, the partial ordering of the latter is denoted by $\leqslant^{\mathfrak{A}}$.

Recall that $\mu^+ \triangleq (\wedge\langle\vee\langle x_j\rangle_{j \in (3\setminus\{i\})}\rangle_{i \in 3}) \in \mathrm{Tm}^3_{\Sigma^+}$ is a majority term for the variety of lattices (cf., e.g., [17] and its bibliography). Therefore, any expansion of a lattice is congruence-distributive.

A *(ternary)[dual]discriminator(term) for* K is any $\tau \in \mathrm{Tm}^3_\Sigma$ such that, for each $\mathfrak{A} \in \mathsf{K}$ and all $\bar{a} \in A^3$, it holds that:

$$\tau^{\mathfrak{A}}[x_i/a_i]_{i \in 3} = \begin{cases} a_2 & \text{if } a_0 = [\neq] a_1, \\ a_0 & \text{otherwise,} \end{cases}$$

in which case $\mathfrak{A}$, being non-one-element, is simple, because, for every $\theta \in (\mathrm{Con}(\mathfrak{A}) \setminus \{\Delta_A\})$, any $\langle a, b\rangle \in (\theta \setminus \Delta_A) \neq \varnothing$ and each $c \in A$, it holds that $a[c] = \tau^{\mathfrak{A}}(a, b, c)\ \theta\ \tau^{\mathfrak{A}}(a, a, c) = c[a]$, and so $\theta = A^2$, while $\varpi[x_1/\tau]$, where ϖ is a permutation term for K, is, conversely, a [non-]dual discriminator for K, whereas τ is both a [dual]discriminator for $\mathbf{V}(\varnothing) \cup \mathbf{ISP}^{\mathrm{U}}\mathsf{K}$ and a minority[majority] term for K. (Conversely, in case each member of K is no-more-than-two-element, any minority[majority] term for it is a [dual]discriminator one.) Then, a (quasi)variety Q of Σ-algebras is said to be *[dual]τ-discriminator*, whenever τ is a[dual] discriminator for $\mathrm{SI}_{(\mathsf{Q})}(\mathsf{Q})$, in which case Q is a both semi-simple and arithmetical [resp., congruence-distributive] variety, in view of Theorem 2.13 and Corollary 2.17. In view of (2.2) and Theorem 2.13, this fits well the notion of a discriminator quasivariety accepted in [26] and definitely introduces clarity into Footnote 4) of [26].

3. Preliminary advanced key issues

3.1. **Closure systems with disjunctive basis.** Throughout this subsection, we fix an arbitrary set A. An *abstract binary system over A* is any $\delta : A^2 \to \wp(A)$, in which case, for all $X, Y \subseteq A$, we set $\delta(X,Y) \triangleq (\bigcup \delta[X \times Y])$. Then, a $Z \subseteq A$ is said to be *δ-disjunctive*, provided, for all $a, b \in A$, it holds that $((\{a,b\} \cap Z) \neq \varnothing) \Leftrightarrow (\delta(a,b) \subseteq Z)$, in which case, for all $X, Y \subseteq A$, we have $((X \subseteq Z)|(Y \subseteq Z)) \Leftrightarrow (\delta(X,Y) \subseteq Z)$. Next, a closure operator C over A is said to be *δ-disjunctive*, provided, for all $a, b \in A$ and every $Z \subseteq A$, it holds that

$$C(Z \cup \delta(a,b)) = (C(Z \cup \{a\}) \cap C(Z \cup \{b\})). \tag{3.1}$$

Lemma 3.1. *Let δ be an abstract binary system over A, C a closure operator over A and $\mathcal{B}$ a closure basis of $\operatorname{img} C$. Suppose each element of $\mathcal{B}$ is δ-disjunctive. Then, $(C(Z \cup X) \cap C(Z \cup Y)) = C(Z \cup \delta(X,Y))$, for all $X, Y, Z \subseteq A$. In particular, C is δ-disjunctive.*

Proof. For all $a \in A$, we then have:

$$\begin{aligned}
&(a \in C(Z \cup X) \cap C(Z \cup Y))\\
\Leftrightarrow\ &\forall W \in \mathcal{B} : ((((Z \subseteq W)\&(X \subseteq W)) \Rightarrow (a \in W))\\
&\&(((Z \subseteq W)\&(Y \subseteq W)) \Rightarrow (a \in W)))\\
\Leftrightarrow\ &\forall W \in \mathcal{B} : (((Z \subseteq W)\&((X \subseteq W)|(Y \subseteq W))) \Rightarrow (a \in W))\\
\Leftrightarrow\ &\forall W \in \mathcal{B} : (((Z \subseteq W)\&(\delta(X,Y) \subseteq W)) \Rightarrow (a \in W))\\
\Leftrightarrow\ &(a \in C(Z \cup \delta(X,Y))),
\end{aligned}$$

as required. ☐

Theorem 3.2. *Let δ be an abstract binary system over A, C a closure operator over A and $\mathcal{B}$ a closure basis of $\operatorname{img} C$. Suppose each element of $\mathcal{B}$ is δ-disjunctive. Then, the lattice $\operatorname{img} C$ is distributive.*

Proof. Consider any $X, Y, Z \in \operatorname{img} C$. Then, applying Lemma 3.1 twice (the second time — with $\varnothing$ instead of Z), we get $((Z \vee X) \cap (Z \vee Y)) = (C(Z \cup X) \cap C(Z \cup Y)) = C(Z \cup \delta(X,Y)) = C(Z \cup C(\delta(X,Y))) = (Z \vee C(\delta(X,Y))) = (Z \vee (X \cap Y))$, as required. ☐

Proposition 3.3. *Let δ be an abstract binary system over A, C a closure operator over A and $X \in \operatorname{img} C$. Suppose C is δ-disjunctive. Then, X is δ-disjunctive iff it is pair-wise-meet-irreducible in $\operatorname{img} C$, and so it is finitely-meet-irreducible in $\operatorname{img} C$ iff it is δ-disjunctive and proper.*

Proof. First, assume X is not δ-disjunctive. Then, in view of (3.1) with $Z = \varnothing$, there is some $\vec{a} \in (A \setminus X)^2$, in which case, for each $i \in 2$, it holds that $X \neq C(X \cup \{a_i\}) \in \operatorname{img} C$, such that $\delta(\vec{a}) \subseteq X$. Therefore, by (3.1), we have $X = \bigcap_{i \in 2} C(X \cup \{a_i\})$. Hence, X is not pair-wise-meet-irreducible in $\operatorname{img} C$.

Conversely, assume X is not pair-wise-meet-irreducible in $\operatorname{img} C$. Then, there is some $\vec{Y} \in ((\operatorname{img} C) \setminus \{X\})^2$ such that $X = \bigcap_{i \in 2} Y_i$, in which case, for each $i \in 2$, $X \subsetneq Y_i$, so there is some $b_i \in (Y_i \setminus X) \neq \varnothing$. In this way, by (3.1), we have $\delta(\vec{b}) \subseteq C(X \cup \delta(\vec{b})) = \bigcap_{i \in 2} C(X \cup \{b_i\}) \subseteq \bigcap_{i \in 2} Y_i = X$. Thus, X is not δ-disjunctive, as required. ☐

As an important instance of closure system with disjunctive basis, we have:

Example 3.4. Let $\mathcal{D}$ be a distributive lattice with zero. Then, prime ideals of it are exactly proper $\wedge$-disjunctive ideals of it. Hence, by the Prime Ideal Theorem, due to which the set of all prime ideals of $\mathcal{D}$ is a basis of the closure system of all ideals of $\mathcal{D}$, and Theorem 3.2, we *immediately* see that the lattice of all ideals of $\mathcal{D}$ is distributive. ☐

3.1.1. *Application to relatively filtral quasivarieties.*

Lemma 3.5. *Let I be a set, $\mathcal{F}$ and $\mathcal{G}$ filters on it, $\overline{\mathfrak{A}}$ an I-tuple of non-one-element Σ-algebras and $\mathfrak{B}$ the direct product of it. Then, $(\mathcal{F} \subseteq \mathcal{G}) \Leftrightarrow (\theta^B_{\mathcal{F}} \subseteq \theta^B_{\mathcal{G}})$.*

Proof. The metaimplication from left to right is immediate. Conversely, assume $\theta^B_{\mathcal{F}} \subseteq \theta^B_{\mathcal{G}}$. Consider any $X \in \mathcal{F}$. Then, for each $i \in (I \setminus X)$, $|A_i| > 1$, so there is some $\vec{a}^i \in (A_i^2 \setminus \Delta_{A_i}) \neq \varnothing$. Moreover, for each $i \in X$, there is some $b_i \in A_i \neq \varnothing$. Define $\vec{c} \in B^2$ as follows:

$$c^j_i \triangleq \begin{cases} b_i & \text{if } i \in X,\\ a^i_j & \text{otherwise,}\end{cases}$$

for all $j \in 2$ and all $i \in I$. Then, $E(\vec{c}^0, \vec{c}^1) = X \in \mathcal{F}$, in which case $\vec{c} \in \theta^B_{\mathcal{F}} \subseteq \theta^B_{\mathcal{G}}$, and so $X \in \mathcal{G}$, as required. ☐

Theorem 3.6. *Any [relatively] (sub)directly filtral [quasi]variety Q is [relatively](sub)directly congruence-distributive(, and so [relatively] congruence-distributive).*

Proof. Consider any set I and any $\overline{\mathfrak{B}} \in \mathrm{SI}_{[\mathsf{Q}]}(\mathsf{Q})^I$(as well as any subdirect product $\mathfrak{C}$ of $\overline{\mathfrak{B}}$). Put $\mathfrak{D} \triangleq \prod_{i \in I} \mathfrak{B}_i$, in which case $(\mathfrak{C},)\mathfrak{D} \in \mathsf{Q}$. Then, any [$\mathsf{Q}$-]congruence of(either $\mathfrak{C}$ or) $\mathfrak{D}$ is filtral[, while, conversely, any filtral congruence of(either $\mathfrak{C}$ or) $\mathfrak{D}$ is Q-relative(, in view of Lemma 2.8 with $h = \Delta_C \in \hom(\mathfrak{C}, \mathfrak{D})$)]. Hence, in particular, by Lemma 3.5, the mapping $\mathcal{F} \mapsto \theta^D_{\mathcal{F}}$ is an isomorphism between the closure systems of all filters on I and $\mathrm{Con}_{[\mathsf{Q}]}(\mathfrak{D})$, in which case, by the dual version of Example 3.4, $\mathrm{Con}_{[\mathsf{Q}]}(\mathfrak{D})$ is distributive.(Moreover, $\mathrm{Con}_{[\mathsf{Q}]}(\mathfrak{C}) = \{\theta \cap C^2 | \theta \in \mathrm{Con}_{[\mathsf{Q}]}(\mathfrak{D})\}$, in which case we have:

$$\mathrm{Cg}^{\mathfrak{C}}_{[\mathsf{Q}]}(X) = (\mathrm{Cg}^{\mathfrak{D}}_{[\mathsf{Q}]}(X) \cap C^2), \tag{3.2}$$

for all $X \subseteq C^2$. Next, consider any $X, Y, Z \in \mathrm{Con}_{[\mathsf{Q}]}(\mathfrak{C})$. Then, by (3.2) and the distributivity of $\mathrm{Con}_{[\mathsf{Q}]}(\mathfrak{D})$, we get:

$$\begin{aligned}
&(X \cap (Y \vee Z)) = (\mathrm{Cg}^{\mathfrak{C}}_{[\mathsf{Q}]}(X) \cap \mathrm{Cg}^{\mathfrak{C}}_{[\mathsf{Q}]}(Y \cup Z)) = \\
&(C^2 \cap (\mathrm{Cg}^{\mathfrak{D}}_{[\mathsf{Q}]}(X) \cap \mathrm{Cg}^{\mathfrak{D}}_{[\mathsf{Q}]}(Y \cup Z))) = \\
&(C^2 \cap (\mathrm{Cg}^{\mathfrak{D}}_{[\mathsf{Q}]}(X) \cap \mathrm{Cg}^{\mathfrak{D}}_{[\mathsf{Q}]}(\mathrm{Cg}^{\mathfrak{D}}_{[\mathsf{Q}]}(Y) \cup \mathrm{Cg}^{\mathfrak{D}}_{[\mathsf{Q}]}(Z)) = \\
&(C^2 \cap (\mathrm{Cg}^{\mathfrak{D}}_{[\mathsf{Q}]}(X) \cap (\mathrm{Cg}^{\mathfrak{D}}_{[\mathsf{Q}]}(Y) \vee \mathrm{Cg}^{\mathfrak{D}}_{[\mathsf{Q}]}(Z)))) = \\
&(C^2 \cap ((\mathrm{Cg}^{\mathfrak{D}}_{[\mathsf{Q}]}(X) \vee \mathrm{Cg}^{\mathfrak{D}}_{[\mathsf{Q}]}(Y)) \cap (\mathrm{Cg}^{\mathfrak{D}}_{[\mathsf{Q}]}(X) \vee \mathrm{Cg}^{\mathfrak{D}}_{[\mathsf{Q}]}(Z)))) = \\
&(C^2 \cap (\mathrm{Cg}^{\mathfrak{D}}_{[\mathsf{Q}]}(\mathrm{Cg}^{\mathfrak{D}}_{[\mathsf{Q}]}(X) \cup \mathrm{Cg}^{\mathfrak{D}}_{[\mathsf{Q}]}(Y)) \cap \mathrm{Cg}^{\mathfrak{D}}_{[\mathsf{Q}]}(\mathrm{Cg}^{\mathfrak{D}}_{[\mathsf{Q}]}(X) \cup \mathrm{Cg}^{\mathfrak{D}}_{[\mathsf{Q}]}(Z)))) = \\
&(C^2 \cap (\mathrm{Cg}^{\mathfrak{D}}_{[\mathsf{Q}]}(X \cup Y) \cap \mathrm{Cg}^{\mathfrak{D}}_{[\mathsf{Q}]}(X \cup Z))) = \\
&(\mathrm{Cg}^{\mathfrak{C}}_{[\mathsf{Q}]}(X \cup Y) \cap \mathrm{Cg}^{\mathfrak{C}}_{[\mathsf{Q}]}(X \cup Z)) = ((X \vee Y) \cap (X \vee Z)).
\end{aligned}$$

Thus, $\mathrm{Con}_{[\mathsf{Q}]}(\mathfrak{C})$ is distributive, in which case Lemma 2.8 and Theorem 2.13 complete the argument.) □

This provides, perhaps, the most transparent insight into the issue of[relative] congruence-distributivity of[relatively] subdirectly filtral [quasi]varieties (by the way, raised and remained open in [3]) not involving REDP[R]C as well as implicative/disjunctive systems for them *at all*. And what is more, meanwhile, it remains a unique way of proving [relative]direct congruence-distributivity of parameterized implicative [quasi]varieties.

3.2. **Disjunctive/implicative systems for algebras.** A *congruence* Σ*-scheme of rank* $\alpha \in (\infty \setminus 4)$ is any $\mho \subseteq \mathrm{Eq}^{\alpha}_{\Sigma}$. This is said to be *restricted/finitary*, whenever it is of rank 4/is both finite and of finite rank.

Remark 3.7. Given any finite congruence Σ-scheme $\mho$ of rank $\alpha \in (\infty \setminus 4)$, the set $U \subseteq V_\alpha$ of all variables occurring in $\mho$ is finite, and so is $V \triangleq (U \setminus V_4)$, in which case there is a bijection $e : V \to m \triangleq |V| \in \omega$. In this way, $\mho' \triangleq (\mho[v/x_{4+e(v)}]_{v \in V})$ is a finitary congruence Σ-scheme of rank $n \triangleq (m+4) \in (\omega \setminus 4)$ such that the sentence $\forall_4((\exists_{\alpha \setminus 4} \bigwedge \mho) \leftrightarrow (\exists_{n \setminus 4} \bigwedge \mho'))$ is a tautology. □

According to [24] and [26] for the restricted case, a *[restricted|finitary]disjunctive/implicative system for* a class K of Σ-algebras is any [restricted|finitary] congruence Σ-scheme of rank $\alpha \in (\infty \setminus 4)$ such that, for each $\mathfrak{A} \in \mathsf{K}$ and all $\bar{a} \in A^4$, it holds that

$$((a_0 \neq / = a_1) \Rightarrow (a_2 = a_3)) \Leftrightarrow (\mathfrak{A} \models (\exists_{\alpha \setminus 4} \bigwedge \mho)[x_i/a_i]_{i \in 4}), \tag{3.3}$$

in which case it is so for $\mathbf{V}(\varnothing) \cup \mathbf{I}[\mathbf{S}|\mathbf{P}^{\mathrm{U}}]\mathsf{K}$, while $\mathfrak{A}$ is said to be $\mho$*-disunctive/-implicative*. Then, a (quasi)variety Q of Σ-algebras is said to be *[restricted|finitely]*$\mho$*-disjunctive/-implicative*, whenever $\mho$ is a [restricted|finitary]disjunctive/implicative system for $\mathrm{SI}_{(\mathsf{Q})}(\mathsf{Q})$. (Within any context, "parameterized" means "non-restricted".) Then, according to Remark 2.4 of [26], disjunctive finite restricted systems are definable via implicative ones.

Remark 3.8. In case $\mathbf{V}(\mathsf{K})$ is locally-finite, the following hold:

(i) a restricted congruence Σ-scheme is a disjunctive/implicative system for K iff it is so for $(\mathbf{S}\mathsf{K})_{<\omega}$;
(ii) in particular, by Lemma 2.18, a[restricted] congruence Σ-scheme is a disjunctive/implicative system for K if[f] it is so for $\mathbf{P}^{\mathrm{U}}\mathsf{K}$. □

3.2.1. *Restricted disjunctivity versus relative congruence distributivity.* Let $\mho \subseteq \mathrm{Eq}^4_{\Sigma}$ and $\mathfrak{A}$ a Σ-algebra. Given any $h : V_4 \to A$, put $\mho^{\mathfrak{A}}[h] \triangleq \{\langle \phi^{\mathfrak{A}}[h], \psi^{\mathfrak{A}}[h] \rangle \mid \langle \phi, \psi \rangle \in \mho\} \subseteq A^2$. In this way, we have the mapping $\delta^{\mathfrak{A}}_{\mho} : (A^2)^2 \to \wp(A^2), \langle \vec{a}, \vec{b} \rangle \mapsto \mho^{\mathfrak{A}}[x_i/a_i; x_{2+i}/b_i]_{i \in 2}$, in which case $\mathfrak{A}$ is $\mho$-disjunctive iff Δ_A is $\delta^{\mathfrak{A}}_{\mho}$-disjunctive.

Lemma 3.9. *Let* $\mho \subseteq \mathrm{Eq}^4_{\Sigma}$*,* Q *a* $\mho$*-disjunctive [quasi]variety of* Σ*-algebras and* $\mathfrak{A} \in \mathsf{Q}$*. Then,* $\mathrm{Con}_{[\mathsf{Q}]}(\mathfrak{A})$ *has a basis consisting of* $\delta^{\mathfrak{A}}_{\mho}$*-disjunctive sets.*

Proof. In that case, by Lemma 2.8, for every $\theta \in \mathrm{MI}(\mathrm{Con}_{[\mathsf{Q}]}(\mathfrak{A}))$, $(\mathfrak{A}/\theta) \in \mathrm{SI}_{[\mathsf{Q}]}(\mathsf{V})$ is $\mho$-disjunctive, in which case θ is $\delta^{\mathfrak{A}}_{\mho}$-disjunctive. In this way, Remark 2.1 and Corollary 2.12 complete the argument. □

Lemma 3.9 and Theorem 3.2 immediately yield:

Theorem 3.10. *Any restricted disjunctive [quasi]variety is [relatively]congruence-distributive.*

Likewise, Lemmas 3.1, 3.9 and Proposition 3.3 imply:

Proposition 3.11. *Let* $\mho \subseteq \mathrm{Eq}^4_{\Sigma}$*,* Q *a* $\mho$*-disjunctive [quasi]variety of* Σ*-algebras and* $\mathfrak{A} \in \mathsf{Q}$*. Then,* $\mathfrak{A}$ *is [*Q*-]finitely-subdirectly-irreducible iff it is* $\mho$*-disjunctive and non-one-element. In particular,* $\mathrm{SI}^{\omega}_{[\mathsf{Q}]}(\mathsf{Q})$ *is the class of all* $\mho$*-disjunctive non-one-element members of* Q*.*

Corollary 3.12. *Let* K *be a [finite]class of [finite]*Σ*-algebras,* $\mho \subseteq \mathrm{Eq}^4_\Sigma$*,* $\mathsf{Q} \triangleq \mathbf{QV}(\mathsf{K})[= \mathbf{PV}(\mathsf{K})]$ *and* $K \subseteq \infty$*. Suppose* $\mho$ *is a disjunctive system for* $\mathbf{P}^{\mathrm{U}}\mathsf{K}$ *(in particular, both* $\mho$ *is a disjunctive system for* K *and either* $\mho$ *is finite or* Q *is locally finite {in particular, both* K *and all members of it are finite}; cf. Remark 3.8(ii)) and* $\{0,2\} \subseteq K$*. Then,* Q *is* $\mho$*-disjunctive, while* $\mathrm{SI}^K_{\mathsf{Q}}(\mathsf{Q}) \subseteq [= \mathbf{IS}_{>1}\mathsf{K} =]\mathbf{IS}_{>1}\mathbf{P}^{\mathrm{U}}\mathsf{K} = \mathrm{SI}^\omega_{\mathsf{Q}}(\mathsf{Q})$*.*

Proof. Then, $\mho$ is a disjunctive system for $\mathbf{ISP}^{\mathrm{U}}\mathsf{K}$. In particular, by (2.2), Q is $\mho$-disjunctive, in which case, by Theorem 3.10, it is relatively congruence-distributive, and so, by Corollary 2.16, we have $\mathrm{SI}^{K/\omega}_{\mathsf{Q}}(\mathsf{Q}) \subseteq \mathbf{IS}_{>1}\mathbf{P}^{\mathrm{U}}\mathsf{K}$. Conversely, by Proposition 3.11, we eventually get $\mathbf{IS}_{>1}\mathbf{P}^{\mathrm{U}}\mathsf{K}[= \mathbf{IS}_{>1}\mathsf{K}] \subseteq \mathrm{SI}^\omega_{\mathsf{Q}}(\mathsf{Q})_{[<\omega]}[\subseteq \mathrm{SI}_{\mathsf{Q}}(\mathsf{Q}) \subseteq \mathrm{SI}^K_{\mathsf{Q}}(\mathsf{Q})]$, as required. ☐

Theorem 3.13. *Let* Q *be a [quasi]variety. Then, the following are equivalent:*

(i) Q *is(finitely)restricted disjunctive;*
(ii) Q *is[relatively] congruence-distributive, while* $\mathrm{SI}^\omega_{[\mathsf{Q}]}(\mathsf{Q}) \cup \mathbf{V}(\varnothing)$ *is a universal(first-order) model class;*
(iii) Q *is[relatively] congruence-distributive, while* $\mathrm{SI}^\omega_{[\mathsf{Q}]}(\mathsf{Q}) \cup \mathbf{V}(\varnothing)$ *is a universal model class(, whereas* $\mathrm{SI}^\omega_{[\mathsf{Q}]}(\mathsf{Q})$ *is a first-order model class);*
(iv) Q *is[relatively] congruence-distributive, while* $\mathrm{SI}^\omega_{[\mathsf{Q}]}(\mathsf{Q})$ *is closed under* $\mathbf{S}_{>1}$*(and* $\mathbf{P}^{\mathrm{U}}$*).*

Proof. First, (i)⇒(ii) is by Theorem 3.10, Proposition 3.11 and the fact that quasivarieties are universal first-order model classes. Next, (ii)⇒(iii) is by the fact that [Q-]finitely-subdirectly-irreducibles of Q are not one-element and the fact that the class of all non-one-element Σ-algebras is axiomatized by the first-order sentence $\Phi_{>1}$. Further, (iii)⇒(iv) is immediate(, with using [13]). Finally, assume (iv) holds. Then, for proving (i), it suffices to argue existence of a (finite)restricted disjunctive system for $\mathsf{K} \triangleq \mathrm{SI}^\omega_{[\mathsf{Q}]}(\mathsf{Q}) \supseteq \mathrm{SI}_{[\mathsf{Q}]}(\mathsf{Q})$.

Put $\mathfrak{T} \triangleq \mathfrak{Tm}^4_\Sigma$, $\theta \triangleq \mathrm{Cg}^{\mathfrak{T}}_{\mathsf{Q}}(\varnothing) \in \mathrm{Con}_{\mathsf{Q}}(\mathfrak{T})$, $\mathfrak{F} \triangleq (\mathfrak{T}/\theta) \in \mathsf{Q}$, $\Phi \triangleq \langle x_0, x_1\rangle \in \mathrm{Eq}^4_\Sigma$, $\Psi \triangleq \langle x_2, x_3\rangle \in \mathrm{Eq}^4_\Sigma$, $a \triangleq \mathrm{Cg}^{\mathfrak{F}}_{[\mathsf{Q}]}(\nu_\theta(\Phi)) \in \mathrm{Con}_{[\mathsf{Q}]}(\mathfrak{F})$ and $b \triangleq \mathrm{Cg}^{\mathfrak{F}}_{[\mathsf{Q}]}(\nu_\theta(\Psi)) \in \mathrm{Con}_{[\mathsf{Q}]}(\mathfrak{F})$. Finally, set $\mho \triangleq \nu_\theta^{-1}[a \cap b] \subseteq \mathrm{Eq}^4_\Sigma$.

Consider any $\mathfrak{A} \in \mathsf{K}$ and any $\bar{a} \in A^4$, in which case $h \triangleq [x_i/a_i]_{i\in 4}$ is extended to the equally-denoted homomorphism from $\mathfrak{T}$ to $\mathfrak{A}$, and so, by Lemma 2.8, $(\ker h) = h^{-1}[\Delta_A] \in \mathrm{Con}_{\mathsf{Q}}(\mathfrak{T})$, for $\Delta_A \in \mathrm{Con}_{\mathsf{Q}}(\mathfrak{A})$, in which case $\theta \subseteq (\ker h)$. In case $\mathrm{img}\, h$ is one-element, (3.3) clearly holds. Otherwise, $\mathfrak{B} \triangleq (\mathfrak{A}{\restriction}(\mathrm{img}\, h)) \in \mathsf{K}$, while $h \in \hom(\mathfrak{T}, \mathfrak{B})$ is surjective. Therefore, by the Homomorphism Theorem, $g \triangleq (h \circ \nu_\theta^{-1}) \in \hom(\mathfrak{F}, \mathfrak{B})$ is surjective, in which case $h = (g \circ \nu_\theta)$, while, by Lemma 2.8, $\vartheta \triangleq (\ker g) = g^{-1}[\Delta_B] \in \mathrm{MI}^\omega(\mathrm{Con}_{[\mathsf{Q}]}(\mathfrak{F}))$, for $\mathfrak{B} \in \mathsf{K}[\subseteq \mathsf{Q}]$, and so $(\ker h) = \nu_\theta^{-1}[\vartheta]$, while, by the distributivity of $\mathrm{Con}_{[\mathsf{Q}]}(\mathfrak{F})$, we have $((a \cap b) \subseteq \vartheta) \Leftrightarrow (\vartheta = (\vartheta \vee (a \cap b) = ((\vartheta \vee a) \cap (\vartheta \vee b))) \Leftrightarrow (\vartheta = (\vartheta \vee a)|\vartheta = (\vartheta \vee b)) \Leftrightarrow (a \subseteq \vartheta | b \subseteq \vartheta)$. In this way, by the surjectivity of $\nu_\theta \in \hom(\mathfrak{T}, \mathfrak{F})$ and Lemma 2.8, we eventually get $(\Phi \in \ker h | \Psi \in \ker h) \Leftrightarrow (\nu_\theta(\Phi) \in \vartheta | \nu_\theta(\Psi) \in \vartheta) \Leftrightarrow (a \subseteq \vartheta | b \subseteq \vartheta) \Leftrightarrow ((a \cap b) \subseteq \vartheta) \Leftrightarrow (\mathfrak{A} \models \mho[h])$. Thus, $\mho$ is a disjunctive system for $\mathfrak{A}$, and so for K.(In particular, by the metaimplication from right to left in (3.3), the Σ-clause $\mho \to \{x_0 \approx x_1, x_2 \approx x_3\}$ is true in K. Hence, by Theorem 2.6, there is some $\mho' \in \wp_\omega(\mho)$ such that the first-order Σ-clause $\mho' \to \{x_0 \approx x_1, x_2 \approx x_3\}$ is true in K. Therefore, $\mho' \subseteq \mho$ is a disjunctive system for K.) Thus, (i) holds, as required. ☐

As an immediate consequence of Theorem 3.13, we first have:

Corollary 3.14. *Let* Q *be a restricted disjunctive [quasi]variety. Then, the following are equivalent:*

(i) Q *is finitely restricted disjunctive;*
(ii) $\mathrm{SI}^\omega_{[\mathsf{Q}]}(\mathsf{Q})$ *is a first-order model class;*
(iii) $\mathrm{SI}^\omega_{[\mathsf{Q}]}(\mathsf{Q})$ *is closed under* $\mathbf{P}^{\mathrm{U}}$*.*

Corollary 3.15. *Let* Q *be a quasivariety of* Σ*-algebras. Then,* Q *is finitely restricted disjunctive iff it is relatively congruence-distributive and generated by some* $\mathsf{K} \subseteq \mathsf{Q}$ *such that* $\mathbf{S}_{>1}\mathbf{P}^{\mathrm{U}}\mathsf{K} \subseteq \mathrm{SI}^M_{\mathsf{Q}}(\mathsf{Q})$*, for some* $\{0,2\} \subseteq M \subseteq \infty$*, in which case:*

$$\mathrm{SI}^K_{\mathsf{Q}}(\mathsf{Q}) \subseteq \mathbf{IS}_{>1}\mathbf{P}^{\mathrm{U}}\mathsf{K} = \mathrm{SI}^M_{\mathsf{Q}}(\mathsf{Q}) = \mathrm{SI}^\omega_{\mathsf{Q}}(\mathsf{Q}), \tag{3.4}$$

for all $\{0,2\} \subseteq K \subseteq \infty$*.*

Proof. The "only if" part is by Theorems 2.13 with $K = \omega$ and 3.13(i)⇒(iv), when taking $M = \omega$ and $\mathsf{K} = \mathrm{SI}^\omega_{\mathsf{Q}}(\mathsf{Q})$. Conversely, assume Q is relatively congruence-distributive and generated by some $\mathsf{K} \subseteq \mathsf{Q}$ such that $\mathbf{S}_{>1}\mathbf{P}^{\mathrm{U}}\mathsf{K} \subseteq \mathrm{SI}^M_{\mathsf{Q}}(\mathsf{Q})$, for some $\{0,2\} \subseteq M \subseteq \infty$. Then, by Corollaries 2.10 and 2.16, we have $\mathrm{SI}^{K/\omega}_{\mathsf{Q}}(\mathsf{Q}) \subseteq \mathbf{IS}_{>1}\mathbf{P}^{\mathrm{U}}\mathsf{K} \subseteq \mathbf{I}\,\mathrm{SI}^M_{\mathsf{Q}}(\mathsf{Q}) \subseteq \mathrm{SI}^M_{\mathsf{Q}}(\mathsf{Q}) \subseteq \mathrm{SI}^\omega_{\mathsf{Q}}(\mathsf{Q})$, where $\{0,2\} \subseteq K \subseteq \infty$, in which case we get (3.4), and so, by it, we conclude that $\mathrm{SI}^\omega_{\mathsf{Q}}(\mathsf{Q})$ is closed under both $\mathbf{S}_{>1}$ and, by Corollary 2.7, $\mathbf{P}^{\mathrm{U}}$, for the class of all non-one-element Σ-algebras is axiomatized by $\Phi_{>1}$, while first-order model classes are closed under $\mathbf{P}^{\mathrm{U}}$ (cf., e.g., [13]). In this way, Theorem 3.13(iv)⇒(i) completes the argument. ☐

In this way, combining Corollary 3.15 with [17] (more specifically, the congruence-distributivity of lattice expansions), we get the following valuable particular case:

Corollary 3.16. *Let* K *be a class of* Σ*-algebras and* $(K \cup M) \subseteq \infty$*. Suppose* $\mathsf{V} \triangleq \mathbf{QV}(\mathsf{K})$ *is a variety,* $\Sigma^+ \subseteq \Sigma$*, each member of* $\mathsf{K}{\restriction}\Sigma^+$ *is a lattice,* $\mathbf{S}_{>1}\mathbf{P}^{\mathrm{U}}\mathsf{K} \subseteq \mathrm{SI}^M(\mathsf{V})$ *and* $\{0,2\} \subseteq (K \cap M)$*. Then, the following hold:*

(i) $\mathrm{SI}^K(\mathsf{V}) \subseteq \mathbf{IS}_{>1}\mathbf{P}^{\mathrm{U}}\mathsf{K} = \mathrm{SI}^M(\mathsf{V}) = \mathrm{SI}^\omega(\mathsf{V})$*;*
(ii) V *is finitely restricted disjunctive.*

3.2.1.1. Restricted disjunctivity versus local finiteness.

Lemma 3.17. *Let* Q *be a locally-finite quasivariety and* $\mathfrak{A} \in (\mathsf{Q} \setminus \mathbf{V}(\varnothing))$*. Suppose* $(\mathbf{S}_{>1}\mathfrak{A})_{<\omega} \subseteq \mathrm{SI}^{\omega}_{\mathsf{Q}}(\mathsf{Q})$*. Then,* $\mathfrak{A} \in \mathrm{SI}^{\omega}_{\mathsf{Q}}(\mathsf{Q})$.

Proof. By contradiction. For suppose $\mathfrak{A} \notin \mathsf{K} \triangleq \mathrm{SI}^{\omega}_{\mathsf{Q}}(\mathsf{Q})$. Then, there are some $n \in \omega$ and some $\bar{\theta} \in (\mathrm{Con}_{\mathsf{Q}}(\mathfrak{A}) \setminus \{\Delta_A\})^n$ such that $\Delta_A = (A^2 \cap \bigcap_{i \in n} \theta_i)$, in which case $n \neq 0$, because $|A| > 1$, and, for each $i \in n$, there is some $\langle a_i, b_i \rangle \in (\theta_i \setminus \Delta_A) \neq \varnothing$. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ generated by $\bigcup_{i \in n}\{a_i, b_i\}$. Then, $\mathfrak{B} \in \mathsf{Q}$ is finitely generated, and so finite. Moreover, as $n \neq 0$ and $a_0 \neq b_0$, $\mathfrak{B}$ is not one-element, in which case $\mathfrak{B} \in \mathsf{K}$. Furthermore, for each $i \in n$, $\langle a_i, b_i \rangle \in \theta_i \in \mathrm{Con}_{\mathsf{Q}}(\mathfrak{A})$, in which case, by Corollary 2.11 with $h = \Delta_B$, $\mathrm{Con}_{\mathsf{Q}}(\mathfrak{B}) \ni \vartheta_i \triangleq \mathrm{Cg}^{\mathfrak{B}}_{\mathsf{Q}}(\langle a_i, b_i \rangle) \subseteq \mathrm{Cg}^{\mathfrak{A}}_{\mathsf{Q}}(\langle a_i, b_i \rangle) \subseteq \theta_i$, and so $\Delta_B \subseteq (B^2 \cap \bigcap_{i \in n} \vartheta_i) \subseteq (B^2 \cap (A^2 \cap \bigcap_{i \in n} \theta_i)) = (B^2 \cap \Delta_A) = \Delta_B$. On the other hand, for each $i \in n$, $a_i \neq b_i$ and $\langle a_i, b_i \rangle \in \vartheta_i$, in which case $\vartheta_i \neq \Delta_B$, and so $\mathfrak{B} \notin \mathsf{K}$, for $\Delta_B \in \mathrm{Con}_{\mathsf{Q}}(\mathfrak{B})$. This contradiction completes the argument. □

Since each member of $\mathrm{SI}^{\omega}_{[\mathsf{Q}]}(\mathsf{Q})$ is not-one-element, i.e., satisfies $\Phi_{>1}$, and so is any ultra-product of them, by Corollaries 2.10, 2.19 with $\mathsf{K} = \mathsf{S} = \mathrm{SI}^{\omega}_{[\mathsf{Q}]}(\mathsf{Q})$ and Lemma 3.17, we immediately get:

Corollary 3.18. *Let* Q *be a locally-finite [quasi]variety. Then,* $\mathrm{SI}^{\omega}_{[\mathsf{Q}]}(\mathsf{Q})$ *is closed under both* $\mathbf{P}^{\mathrm{U}}$ *and* $\mathbf{S}_{>1}$ *iff it is closed under* $\mathbf{S}_{>1}$ *iff* $(\mathbf{S}_{>1}\,\mathrm{SI}^{\omega}_{[\mathsf{Q}]}(\mathsf{Q}))_{<\omega} \subseteq \mathrm{SI}^{\omega}_{[\mathsf{Q}]}(\mathsf{Q})$.

Then, combining Corollary 3.18 with Theorem 3.13(i)⇔(iv), we get:

Corollary 3.19. *Let* Q *be s locally-finite [quasi]variety. Then, the following are equivalent:*

(i) Q *is restricted disjunctive;*
(ii) Q *is restricted finitely disjunctive;*
(iii) Q *is [relatively]congruence-distributive, while* $\mathrm{SI}^{\omega}_{[\mathsf{Q}]}(\mathsf{Q})$ *is closed under* $\mathbf{S}_{>1}$*;*
(iv) Q *is [relatively]congruence-distributive, while* $(\mathbf{S}_{>1}\,\mathrm{SI}^{\omega}_{[\mathsf{Q}]}(\mathsf{Q}))_{<\omega} \subseteq \mathrm{SI}^{\omega}_{[\mathsf{Q}]}(\mathsf{Q})$.

Corollary 3.20. *Let* K *be a[finite] class of[finite]* Σ*-algebras and* $(K \cup L) \subseteq \infty$*. Suppose* $\mathsf{Q} \triangleq \mathbf{QV}(\mathsf{K})[= \mathbf{PV}(\mathsf{K})]$ *is both locally finite {in particular, both* K *and all members of it are finite} and relatively congruence-distributive,* $(\mathbf{S}_{>1}\mathsf{K})_{<\omega} \subseteq \mathrm{SI}^{L}_{\mathsf{Q}}(\mathsf{Q})$ *and* $\{0, 2\} \subseteq (K \cap L)$*. Then, the following hold:*

(i) $\mathrm{SI}^{K}_{\mathsf{Q}}(\mathsf{Q}) \subseteq [=]\,\mathrm{SI}^{\omega}_{\mathsf{Q}}(\mathsf{Q}) = \mathbf{IS}_{>1}\mathbf{P}^{\mathrm{U}}\mathsf{K}[= \mathbf{IS}_{>1}\mathsf{K}]$*;*
(ii) Q *is finitely restricted disjunctive.*

Proof. By Corollaries 2.10, 3.15 with $M = \omega$, 2.19 with $\mathsf{S} = \mathrm{SI}^{\omega}_{\mathsf{Q}}(\mathsf{Q})$, Lemma 3.17 and the inclusion $\mathrm{SI}^{L}_{\mathsf{Q}}(\mathsf{Q}) \subseteq \mathrm{SI}^{\omega}_{\mathsf{Q}}(\mathsf{Q})$, we immediately get both (ii) and (i) but without the first optional equality "[=]".[Finally, each member of $\mathrm{SI}^{\omega}_{\mathsf{Q}}(\mathsf{Q}) = \mathbf{IS}_{>1}\mathsf{K}$ is finite, in which case it is Q-subdirectly-irreducible, and so Q-K-subdirectly-irreducible, as required.] □

In this way, combining Corollary 3.20 with [17] (more specifically, the congruence-distributivity of lattice expansions), we get the following valuable generic result covering, in particular, the variety of Stone algebras (cf. Subsection 6.4):

Corollary 3.21. *Let* K *be a[finite] class of[finite]* Σ*-algebras and* $(K \cup M) \subseteq \infty$*. Suppose* $\mathsf{V} \triangleq \mathbf{QV}(\mathsf{K})$ *is both locally finite {in particular, both* K *and all members of it are finite} and a variety,* $\Sigma^{+} \subseteq \Sigma$*, each member of* $\mathsf{K}{\upharpoonright}\Sigma^{+}$ *is a lattice,* $(\mathbf{S}_{>1}\mathsf{K})_{<\omega} \subseteq \mathrm{SI}^{M}(\mathsf{V})$ *and* $\{0, 2\} \subseteq (K \cap M)$*. Then, the following hold:*

(i) $\mathrm{SI}^{K}(\mathsf{V}) \subseteq [=]\,\mathrm{SI}^{\omega}(\mathsf{V}) = \mathbf{IS}_{>1}\mathbf{P}^{\mathrm{U}}\mathsf{K}[= \mathbf{IS}_{>1}\mathsf{K}]$*;*
(ii) V *is finitely restricted disjunctive.*

3.2.2. *Implicative systems versus congruence schemes.* A congruence Σ-scheme $\mho$ of rank $\alpha \in (\infty \setminus 4)$ is said to be that *for* a class of Σ-algebras K, provided the Σ-implication

$$(\{x_0 \approx x_1\} \cup \mho) \to (x_2 \approx x_3) \tag{3.5}$$

of rank α is true in K, in which case it is so for $\mathbf{PV}(\mathsf{K})$.

Remark 3.22. In view of Theorem 2.6, any congruence Σ-scheme for K includes a finite one, whenever K is closed under $\mathbf{P}^{\mathrm{U}}$. □

Remark 3.23. The following hold:

(i) the metaimplication from right to left in (3.3) in the implicative case holds *for all* $\mathfrak{A} \in \mathsf{K}$ and $\bar{a} \in A^4$ iff $\mho$ is a congruence Σ-scheme for K;
(ii) in particular, by Theorem 2.13, $\mho$ is a congruence Σ-scheme for any $\mho$-implicative quasivariety;
(iii) therefore, by Remarks 3.22 and 3.7, any [restricted]implicative quasivariety is finitely so;[6]
(iv) in particular, in view of (2.2) and Theorem 2.13, a quasivariety is restricted implicative iff it is implicative in the sense of [26];
(v) and what is more, by Remark 2.4 of [26], restricted implicative quasivarieties are finitely restricted disjunctive. □

Lemma 3.24. *Let* K *be a class of* Σ*-algebras,* $\mho$ *a congruence* Σ*-scheme of rank* $\alpha \in (\infty \setminus 4)$ *for* K *and* $\mathfrak{A}$ *a non-one-element* $\mho$*-implicative* Σ*-algebra. Then,* $\mathfrak{A}$ *is* K*-simple.*

[6]This fact is often used tacitly throughout the rest of the paper.

Proof. Consider any $\theta \in (\mathrm{Con}_{\mathsf{K}}(\mathfrak{A}) \setminus \{\Delta_A\})$ and any $a_2, a_3 \in A$. Take any $\langle a_0, a_1\rangle \in (\theta \setminus \Delta_A) \neq \varnothing$. Put $e \triangleq [x_i/a_i]_{i\in 4}$. Then, $\mathfrak{A} \models (\exists_{\alpha\setminus 4} \bigwedge \mho)[e]$, in which case we have $\mathsf{K} \ni (\mathfrak{A}/\theta) \models (\exists_{\alpha\setminus 4} \bigwedge \mho)[\nu_\theta \circ e]$, and so, by Remark 3.23(i), we get $\langle a_2, a_3\rangle \in \theta$. Thus, $\theta = A^2$, as required. □

Remark 3.23(ii) and Lemma 3.24 then yield the following two corollaries:

Theorem 3.25. *Any implicative [quasi]variety is[relatively] semi-simple.*

Theorem 3.26. *Let* Q *be a [quasi]variety of* Σ*-algebras and* $\mho$ *a congruence* Σ*-scheme. Suppose* Q *is* $\mho$*-implicative. Then,* $\mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q})$ *is the class of all non-one-element* $\mho$*-implicative members of* Q*.*

3.2.2.1. Restricted implicativity versus relative congruence-distributivity. First of all, by Remark 3.23(v) and Theorem 3.10, we have:

Theorem 3.27. *Any restricted implicative [quasi]variety is [relatively]congruence-distributive.*

Corollary 3.28. *Let* K *be a [finite]class of [finite]*Σ*-algebras,* $\mho \subseteq \mathrm{Eq}^4_\Sigma$*,* $\mathsf{Q} \triangleq \mathbf{QV}(\mathsf{K})[= \mathbf{PV}(\mathsf{K})]$ *and* $K \subseteq \infty$*. Suppose* $\mho$ *is an implicative system for* $\mathbf{P}^{\mathrm{U}}\mathsf{K}$ *(in particular, both* $\mho$ *is an implicative system for* K *and either* $\mho$ *is finite or* Q *is locally finite {in particular, both* K *and all members of it are finite}; cf. Remark 3.8(ii)) and* $\{0,2\} \subseteq K$*. Then,* Q *is* $\mho$*-implicative, while* $\mathrm{SI}^K_{\mathsf{Q}}(\mathsf{Q}) = \mathrm{Si}_{\mathsf{Q}}(\mathsf{Q}) = \mathbf{IS}_{>1}\mathbf{P}^{\mathrm{U}}\mathsf{K}[= \mathbf{IS}_{>1}\mathsf{K}]$*.*

Proof. Then, $\mho$ is an implicative system for $\mathbf{ISP}^{\mathrm{U}}\mathsf{K}$. In particular, by (2.2), Q is $\mho$-implicative, in which case, by Theorem 3.27, it is relatively congruence-distributive, and so, by Corollary 2.16, we have $\mathrm{Si}_{\mathsf{Q}}(\mathsf{Q}) \subseteq \mathrm{SI}^K_{\mathsf{Q}}(\mathsf{Q}) \subseteq \mathbf{IS}_{>1}\mathbf{P}^{\mathrm{U}}\mathsf{K}[= \mathbf{IS}_{>1}\mathsf{K}]$. Conversely, by Theorem 3.26, we eventually get $\mathbf{IS}_{>1}\mathbf{P}^{\mathrm{U}}\mathsf{K} \subseteq \mathrm{Si}_{\mathsf{Q}}(\mathsf{Q})$, as required. □

Combining Theorem 2.13, Remark 3.23(iii) and the "[]"-option-free case of Corollary 3.28, we get the following interesting consequence generalizing the restricted particular case of Theorem 3.25:

Corollary 3.29. *Let* Q *be a restricted implicative [quasi]variety and* $K \subseteq \infty$*. Suppose* $\{0,2\} \subseteq K$*. Then,* $\mathrm{SI}^K_{[\mathsf{Q}]}(\mathsf{Q}) = \mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q})$*. In particular,* $\mathrm{SI}^\omega_{[\mathsf{Q}]}(\mathsf{Q}) = \mathrm{SI}_{[\mathsf{Q}]}(\mathsf{Q})$*.*

Finally, we have the following practically useful result (cf. Subsections 6.2 and 6.5):

Corollary 3.30. *Let* Q *be a restricted implicative [quasi]variety,* I *a finite set,* $\overline{\mathfrak{A}} \in \mathrm{SI}_{[\mathsf{Q}]}(\mathsf{Q})$*,* $\mathfrak{B} \in \mathbf{S}(\prod_{i\in I}\mathfrak{A}_i)$ *and* $\Theta^B_J \triangleq \ker[\{\pi_j{\upharpoonright}B \mid j \in I\}]$*, where* $J \subseteq I$*. Then,* Θ^B_I *is a basis of* $\mathrm{Con}_{[\mathsf{Q}]}(\mathfrak{B})$*. In particular,* $\mathrm{Con}_{[\mathsf{Q}]}(\mathfrak{B}) = (\theta^B_I \cup \{\Delta_B, B^2\})$*, whenever* $|I| \leqslant 2$*.*

Proof. First, by Lemma 2.8, we have $\Theta^B_I \subseteq \mathrm{Con}_{[\mathsf{Q}]}(\mathfrak{B})$. Next, consider any $\theta \in \mathrm{Con}_{[\mathsf{Q}]}(\mathfrak{B})$. Then, for each $j \in J \triangleq \{i \in I \mid \theta \subseteq \ker(\pi_i{\upharpoonright}B) \neq B^2\} \subseteq I$, $(\theta \vee \ker(\pi_j{\upharpoonright}B)) = \ker(\pi_j{\upharpoonright}B)$, while $\Theta^B_J \subseteq \Theta^B_I$, whereas, for every $i \in (I\setminus J)$, either $\ker(\pi_i{\upharpoonright}B) = B^2$, in which case $(\theta \vee \ker(\pi_i{\upharpoonright}B)) = B^2$, or, otherwise, $\theta \nsubseteq \ker(\pi_i{\upharpoonright}B)$, while, by Theorems 3.25 and 3.26, $\pi_i{\upharpoonright}B$ is a surjective homomorphism from $\mathfrak{B}$ onto $(\mathfrak{A}_i{\upharpoonright}\pi_i[B]) \in \mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q})$, in which case, by Lemma 2.8, we have $\ker(\pi_i{\upharpoonright}B) \in \max(\mathrm{Con}_{[\mathsf{Q}]}(\mathfrak{B})\setminus\{B^2\})$, and so we get $(\theta \vee \ker(\pi_i{\upharpoonright}B)) = B^2$ as well. Moreover, by Theorem 3.27, $\mathfrak{B} \in \mathsf{Q}$ is [Q-]congruence-distributive. Therefore, by the finiteness of I, we eventually get $\theta = (\theta \vee \Delta_B) = (\theta \vee (B^2 \cap \bigcap_{i\in I}\ker(\pi_i{\upharpoonright}B))) = (B^2 \cap \bigcap_{i\in I}(\theta \vee \ker(\pi_i{\upharpoonright}B))) = (B^2 \cap \bigcap \Theta^B_J)$, as required. □

3.2.2.1.1. Restricted implicativity versus equationality.

Theorem 3.31. *A restricted implicative quasivariety* Q *is a variety iff it is both congruence-distributive and semi-simple.*

Proof. The "only if" part is by Theorems 3.25 and 3.27. Conversely, assume Q is both congruence-distributive and semi-simple. Then, by Remark 3.23(iii) and Theorem 3.26, $\mathsf{K} \triangleq \mathrm{Si}_{\mathsf{Q}}(\mathsf{Q}) \subseteq \mathrm{SI}_{\mathsf{Q}}(\mathsf{Q}) \subseteq \mathrm{Si}(\mathsf{Q})$ is closed under $\mathbf{S}_{>1}\mathbf{P}^{\mathrm{U}}$. Moreover, by Theorem 2.13, $\mathsf{Q} = \mathbf{QV}(\mathsf{K})$. In this way, Corollary 2.17 completes the proof. □

The following result has found certain applications to expansions of De Morgan lattices (cf. Subsection 6.2):

Corollary 3.32. *Let* $\Sigma' \subseteq \Sigma'' \subseteq \Sigma$*,* K *a class of* Σ*-algebras and* $\mho \subseteq \mathrm{Eq}^4_{\Sigma'}$*. Suppose either* $\mho$ *is finite or* $\mathbf{V}(\mathsf{K}{\upharpoonright}\Sigma'')$ *is locally-finite (in particular, both* K *and all members of it are finite),* $\mho$ *is an implicative system for* $\mathsf{K}{\upharpoonright}\Sigma'$ *and* $\mathsf{V} \triangleq \mathbf{QV}(\mathsf{K}{\upharpoonright}\Sigma')$ *is a variety. Then, so is* $\mathsf{Q} \triangleq \mathbf{QV}(\mathsf{K})$*.*

Proof. In that case, $\mho$ is an implicative system for $\mathsf{K}{\upharpoonright}\Sigma''$, and so for $\mathbf{P}^{\mathrm{U}}(\mathsf{K}{\upharpoonright}\Sigma'')$, in view of Remark 3.8(ii), in which case it is so for both $\mathbf{P}^{\mathrm{U}}(\mathsf{K}{\upharpoonright}\Sigma')$ and $\mathbf{P}^{\mathrm{U}}\mathsf{K}$. Hence, by Corollary 3.28, $\mathsf{V}[\mathsf{Q}]$ is a restricted $\mho$-implicative [quasi]variety, while every member of $\mathbf{IS}_{>1}\mathbf{P}^{\mathrm{U}}(\mathsf{K}{\upharpoonright}\Sigma')$ is simple, and so is that of $\mathbf{IS}_{>1}\mathbf{P}^{\mathrm{U}}\mathsf{K} = \mathrm{SI}_{\mathsf{Q}}(\mathsf{Q})$, in which case Q is semi-simple. Moreover, by Theorem 3.27, V is congruence-distributive, and so is Q, in view of Theorem 2.3 of [16]. In this way, Theorem 3.31 completes the argument. □

3.2.2.2. Implicativity versus EDPRC. According to [3] for the equational case, a [quasi]variety Q of Σ-algebras is said to have *(restricted)equationally definable principal [relative]congruences* (*(R)EDP[R]C*, for short) *with respect to* a finitary (restricted)congruence Σ-scheme $\mho$ of rank $\alpha \in (\omega \setminus 4)$, provided, for each $\mathfrak{A} \in \mathsf{Q}$ and all $\bar{a} \in A^4$, it holds that

$$(\langle a_2, a_3\rangle \in \mathrm{Cg}^{\mathfrak{A}}_{[\mathsf{Q}]}(\langle a_0, a_1\rangle)) \Leftrightarrow (\mathfrak{A} \models (\exists_{\alpha\setminus 4} \bigwedge \mho)[x_i/a_i]_{i\in 4}). \tag{3.6}$$

Lemma 3.33. *Let* K *be a class of* Σ*-algebras closed under* $\mathbf{IP}^{\mathrm{SD}}$*,* $\mathfrak{A}$ *a* Σ*-algebra and* $\bar{a} \in A^4$*. Suppose* $\mathfrak{A}$ *is in* K*/is either one-element or* K*-simple. Then,* $(\langle a_2, a_3\rangle \in \mathrm{Cg}^{\mathfrak{A}}_{\mathsf{K}}(\langle a_0, a_1\rangle)) \Rightarrow / \Leftarrow ((a_0 = a_1) \Rightarrow (a_2 = a_3))$*.*

Proof. Put $\theta \triangleq \mathrm{Cg}^{\mathfrak{A}}_{\mathsf{K}}(\langle a_0, a_1\rangle) \in \mathrm{Con}_{\mathsf{K}}(\mathfrak{A})$.

First, assume $\mathfrak{A} \in \mathsf{K}$, in which case $\Delta_A \in \mathrm{Con}_{\mathsf{K}}(\mathfrak{A})$, $\langle a_2, a_3\rangle \in \theta$ and $a_0 = a_1$, in which case $\theta = \Delta_A$, and so $a_2 = a_3$.

Conversely, assume $\mathfrak{A}$ is either one-element or K-simple. If $a_0 \neq a_1$, then $\mathfrak{A}$ is not one-element, and so K-simple, in which case θ containing $\langle a_0, a_1\rangle \notin \Delta_A$ is not equal to Δ_A, and so $\theta = A^2 \ni \langle a_2, a_3\rangle$. Likewise, in case $a_2 = a_3$, we have $\langle a_2, a_3\rangle \in \Delta_A \subseteq \theta$, as required. □

By Lemma 3.33 and the metaimplication from right to left in (3.6), we first have:

Corollary 3.34. *Let Q be a [quasi]variety of Σ-algebras and $\mho$ a (restricted)finitary congruence Σ-scheme. Suppose Q has (R)EDP[R]C with respect to $\mho$ for Q. Then, $\mho$ is a congruence Σ-scheme for Q.*

By Lemmas 3.24, 3.33 and Corollary 3.34, we also have:

Corollary 3.35. *Let Q be a [quasi]variety of Σ-algebras, $\mathfrak{A} \in \mathsf{Q}$ and $\mho \in \wp_\omega(\mathrm{Eq}^\alpha_\Sigma)$, where $\alpha \in (\omega \setminus 4)$. Suppose Q has EDP[R]C with respect to $\mho$. Then, $\mathfrak{A}$ is [Q-]simple iff it is $\mho$-implicative and non-one-element.*

Lemma 3.36. *Let $\mho \in \wp_\omega(\mathrm{Eq}^4_\Sigma)$ and Q a $\mho$-implicative [quasi]variety of Σ-algebras. Then, Q has REDP[R]C with respect to $\mho$.*

Proof. Consider any $\mathfrak{A} \in \mathsf{Q}$ and any $\bar{a} \in A^4$, in which case $h \triangleq [x_i/a_i]_{i\in 4}$ is extended to the equally-denoted homomorphism from $\mathfrak{Tm}^4_\Sigma$ to $\mathfrak{A}$. Then, by Remark 2.1 and Corollary 2.12, $\Theta \triangleq \mathrm{MI}(\mathrm{Con}_{[\mathsf{Q}]}(\mathfrak{A}))$ is a closure basis of the inductive closure system $\mathrm{Con}_{[\mathsf{Q}]}(\mathfrak{A})$ over A^2, in which case, in particular, $\Delta_A = (A^2 \cap \bigcap\Theta)$. Moreover, by Lemma 2.8, for every $\theta \in \Theta$, $(\mathfrak{A}/\theta) \in \mathrm{SI}_{[\mathsf{Q}]}(\mathsf{Q})$ is $\mho$-implicative. In this way, we eventually get $(\langle a_2, a_3\rangle \in \mathrm{Cg}^{\mathfrak{A}}_{[\mathsf{Q}]}(\langle a_0, a_1\rangle)) \Leftrightarrow (\forall\theta \in \Theta : ((a_0\ \theta\ a_1) \Rightarrow (a_2\ \theta\ a_3))) \Leftrightarrow (\forall\theta \in \Theta : ((\mathfrak{A}/\theta) \models (\bigwedge\mho)[\nu_\theta \circ h])) \Leftrightarrow (\mho \subseteq (\mathrm{Eq}^4_\Sigma \cap \bigcap_{\theta\in\Theta} \ker(\nu_\theta \circ h)) = (h^{-1}[A^2] \cap \bigcap_{\theta\in\Theta} h^{-1}[\theta]) = h^{-1}[A^2 \cap \bigcap\Theta] = h^{-1}[\Delta_A] = (\ker h)) \Leftrightarrow (\mathfrak{A} \models (\bigwedge\mho)[h])$, as required. □

As an immediate consequence of(Lemma 3.36 and) Corollary 3.35, we then get:

Theorem 3.37. *Let Q be a [quasi]variety of Σ-algebras and $\mho$ a (restricted)finitary congruence Σ-scheme. Then, Q is $\mho$-implicative if(f) it has (R)EDP[R]C with respect to $\mho$ and is [relatively]semi-simple.*

3.2.3. *Disjunctivity/implicativity versus [dual]discriminators.* Given any $\tau \in \mathrm{Tm}^3_\Sigma$, put

$$\begin{aligned}
\mho^{\supset}_\tau &\triangleq \{\tau \approx (\tau[x_2/x_3])\},\\
\mho^{\partial\supset}_\tau &\triangleq \{(\tau[x_0/x_{2+k}, x_1/x_{3-k}, x_2/(\tau[x_2/x_{2+k}])]) \approx x_{2+k} \mid k \in 2\},\\
\mho^{\vee}_\tau &\triangleq \{(\tau[x_0/\tau, x_1/(\tau[x_2/x_3])]) \approx (\tau[x_0/\tau, x_1/(\tau[x_2/x_3]), x_2/x_3])\}.
\end{aligned}$$

Remark 3.38 (cf. Subsection 2.4 of [26] for the non-dual implicative case). Given any [dual]discriminator term $\tau \in \mathrm{Tm}^3_\Sigma$ for a class of Σ-algebras K[, taking the validity of the majority identities (2.3) and $\tau \approx (\tau[x_k/x_{1-k}]_{k\in 2})$ in K into account], $\mho^{\vee/[\partial]\supset}_\tau$ is a finite restricted disjunctive/implicative system for K. In particular, any [dual]τ-discriminator (quasi)variety is restricted finitely $\mho^{\vee/[\partial]\supset}_\tau$-disjunctive/-implicative. □

Remark 3.38 and Lemma 3.36 immediately yield:

Corollary 3.39. *Let $\tau \in \mathrm{Tm}^3_\Sigma$ and Q a[dual]τ-discriminator (quasi)variety of Σ-algebras. Then, Q has REDP(R)C with respect to $\mho^{[\partial]\supset}_\tau$.*

Lemma 3.40. *Let K be a class of Σ-algebras, $\mathfrak{A} \in \mathsf{V} \triangleq \mathbf{V}(\mathsf{K})$, $\mathsf{S} \subseteq \mathrm{Si}(\mathsf{V})$[, $\mho^{\vee/\supset} \in \wp_\omega(\mathrm{Eq}^4_\Sigma)$ a disjunctive/implicative system for K] and $K \subseteq \infty$. Suppose $\mathbf{IS}_{>1}\mathbf{P}^{\mathrm{U}}\mathsf{K} \subseteq \mathsf{S}$ (in particular, $(\mathsf{S} \cup \mathbf{V}(\varnothing)) \supseteq \mathsf{K}$ is closed under $\mathbf{ISP}^{\mathrm{U}}$, that is, a universal first-order model class; cf. Corollary 2.7), V is congruence-distributive (in particular, K has a majority term; cf.* [17]*) and $\{0, 2\} \subseteq K$. Then, the following are equivalent:*

(i) $\mathfrak{A} \in \mathsf{S}$*;*
(ii) $\mathfrak{A}$ *is simple;*
(iii) $\mathfrak{A}$ *is K-subdirectly-irreducible;*
(iv) $\mathfrak{A} \in \mathbf{IS}_{>1}\mathbf{P}^{\mathrm{U}}\mathsf{K}$*[;*
(v) $\mho^{\vee/\supset}$ *is a/an disjunctive/implicative system for $\mathfrak{A}$ and $|A| > 1$.]*

In particular, $\mathsf{V} = \mathbf{QV}(\mathsf{K})$[is restricted finitely $\mho^{\vee/\supset}$-disjunctive/-implicative].

Proof. First, (iv)⇒(i)⇒(ii)⇒(iii) are trivial.[Likewise, (iv)⇒(v) is immediate.] Next, (iii)⇒(iv) is by Corollary 2.15 and (iv)⇒(ii). Further, by Corollary 2.17[and (iii)⇒(v) with $K = \infty$], we conclude that $\mathsf{V} = \mathbf{QV}(\mathsf{K})$[is restricted finitely $\mho^{\vee/\supset}$-disjunctive/-implicative].[Finally, (v)⇒(ii) is then by both (iii)⇒(ii) with $K = \omega$ and Proposition 3.11/Theorem 3.26 alone, respectively]. □

Remark 3.41. In view of Remark 3.38, in case τ is a [dual]discriminator for K, the "[]"-optional case of Lemma 3.40 with S, being the class of all those non-one-element members of V for which τ is a [dual]discriminator, and $\mho^{\vee/\supset} = \mho^{\vee/[\partial]\supset}_\tau$ is well applicable to K. □

4. Equality determinants, equational implications and inequality systems

A *(logical)* Σ-*matrix* (cf. [10]) is any algebraic system $\mathcal{A}$ of the first-order signature $\Sigma \cup \{D\}$ with unary relation symbol D, naturally identified with the couple $\langle\mathfrak{A}, D^{\mathcal{A}}\rangle$. This is said to be $\diamond$-*conjunctive*, where $\diamond \in \Sigma$ is binary, provided $(\{a, b\} \subseteq D^{\mathcal{A}}) \Leftrightarrow ((a \diamond^{\mathfrak{A}} b) \in D^{\mathcal{A}})$, for all $a, b \in A$. Elements of $\mathrm{Con}(\mathcal{A}) \triangleq \{\theta \in \mathrm{Con}(\mathfrak{A}) | \theta[D^{\mathcal{A}}] \subseteq D^{\mathcal{A}}\} \ni \Delta_A$ are called *congruences of* $\mathcal{A}$. Then, $\mathcal{A}$ is said to be *simple*, provided $\mathrm{Con}(\mathcal{A})$ is one-element, that is, contains Δ_A alone.

Given a class M of Σ-matrices and a class K of Σ-algebras, set $(\mathsf{M}{\upharpoonright}\mathsf{K}) \triangleq \{\mathcal{A} \in \mathsf{M} \mid \mathfrak{A} \in \mathsf{K}\}$.

An *equality determinant for* a class M of Σ-matrices is any $\Upsilon \subseteq \mathrm{Tm}^1_\Sigma$ such that, for each $\mathfrak{A} \in (\mathsf{M}{\upharpoonright}\Sigma)$, any $a, b \in A$ are equal, whenever, for every $\mathcal{B} \in (\mathsf{M}{\upharpoonright}\{\mathfrak{A}\})$ and all $\varphi \in \Upsilon$, $\varphi^{\mathfrak{A}}[x_0/a] \in D^{\mathcal{B}}$ iff $\varphi^{\mathfrak{A}}[x_0/b] \in D^{\mathcal{B}}$. (Clearly, Υ is an equality determinant for M, whenever it is so for each member of it. And what is more, in case M consists of a single member, the above definition fits well the original one of [23].)

Likewise, a *[weak]equational implication for* M is any $\varepsilon \subseteq \mathrm{Eq}^2_\Sigma$ such that, for each $\mathfrak{A} \in (\mathsf{M}{\upharpoonright}\Sigma)$ and all $\vec{a} \in A^2$, it holds that:

$$(\mathfrak{A} \models (\bigwedge \varepsilon)[x_i/a_i]_{i \in 2})[\Rightarrow] \Leftrightarrow (\forall \mathcal{B} \in (\mathsf{M}{\upharpoonright}\{\mathfrak{A}\}) : ((a_0 \in D^{\mathcal{B}}) \Rightarrow (a_1 \in D^{\mathcal{B}}))). \tag{4.1}$$

(Clearly, ε is a [weak]equational implication for M if[f] it is so for each member of it. And what is more, in case M consists of a single member, the above definition fits well the original one of Appendix A of [26].)

Proposition 4.1 (cf. Lemma 9 of [24])**.** *Let $\mathcal{A}$ be a Σ-matrix with equality determinant Υ and equational implication ε, $\mathfrak{B}$ a subalgebra of $\mathfrak{A}$ and $h \in \hom(\mathfrak{B}, \mathfrak{A})$. Suppose h is not singular. Then, h is diagonal.*

Proof. In that case, $\mathfrak{C} \triangleq (\mathfrak{B}{\upharpoonright}(\mathrm{img}\, h))$ is a subalgebra of $\mathfrak{A}$. If $(C \cap D^{\mathcal{A}})$ was in $\{\varnothing, C\}$, for all $c, d \in C$ and all $\varphi \in \Upsilon$, we would have $\varphi^{\mathfrak{A}}(c) = \varphi^{\mathfrak{C}}(c) \in D^{\mathcal{A}}$ iff $\varphi^{\mathfrak{A}}(d) = \varphi^{\mathfrak{C}}(d) \in D^{\mathcal{A}}$, in which case we would get $c = d$, and so h would be singular. Therefore, there are some $a, b \in B \subseteq A$ such that $h(a) \notin D^{\mathcal{A}} \ni h(b)$.

Consider any $e \in B$ and any $\upsilon \in \Upsilon$.

First, assume $\upsilon^{\mathfrak{A}}(e) \in D^{\mathcal{A}}$. Then, $\mathfrak{A} \models (\bigwedge \varepsilon)[x_0/b, x_1/\upsilon^{\mathfrak{A}}(e)]$, in which case $\mathfrak{B} \models (\bigwedge \varepsilon)[x_0/b, x_1/\upsilon^{\mathfrak{B}}(e)]$, and so $\mathfrak{A} \models (\bigwedge \varepsilon)[x_0/h(b), x_1/\upsilon^{\mathfrak{A}}(h(e))]$. Hence, $\upsilon^{\mathfrak{A}}(h(e)) \in D^{\mathcal{A}}$.

Conversely, assume $\upsilon^{\mathfrak{A}}(e) \notin D^{\mathcal{A}}$. Then, $\mathfrak{A} \models (\bigwedge \varepsilon)[x_0/\upsilon^{\mathfrak{A}}(e), x_1/a]$, so $\mathfrak{B} \models (\bigwedge \varepsilon)[x_0/\upsilon^{\mathfrak{B}}(e), x_1/a]$, in which case $\mathfrak{A} \models (\bigwedge \varepsilon)[x_0/\upsilon^{\mathfrak{A}}(h(e)), x_1/h(a)]$. Hence, $\upsilon^{\mathfrak{A}}(h(e)) \notin D^{\mathcal{A}}$.

Thus, $(\upsilon^{\mathfrak{A}}(e) \in D^{\mathcal{A}}) \Leftrightarrow (\upsilon^{\mathfrak{A}}(h(e)) \in D^{\mathcal{A}})$, for all $\upsilon \in \Upsilon$, and so $h(e) = e$, as required. □

Let $\Sigma \supseteq \Sigma^+$.

Given a Σ-algebra $\mathfrak{A}$, a non-empty $F \subseteq A$ is called a *filter of* $\mathfrak{A}$, provided $\langle\mathfrak{A}, F\rangle$ is $\wedge$-conjunctive. Then, a proper filter F of $\mathfrak{A}$ is said to be *prime*, provided $\langle\mathfrak{A}, A \setminus F\rangle$ is $\vee$-conjunctive. (This fits well the standard lattice-theoretic terminology.) The set of all prime filters of $\mathfrak{A}$ is denoted by $\mathrm{PF}(\mathfrak{A})$.

Given a class K of Σ-algebras, set $\mathsf{PF}(\mathsf{K}) \triangleq \{\langle\mathfrak{A}, F\rangle \mid \mathfrak{A} \in \mathsf{K}, F \in \mathrm{PF}(\mathfrak{A})\}$. Then, an *equality determinant for* K is any uniform equality determinant for all members of $\mathsf{PF}(\mathsf{K})$.

Put $\varepsilon^+ \triangleq \{x_0 \lessapprox x_1\} \subseteq \mathrm{Eq}^2_\Sigma$.

Remark 4.2. Given a Σ-algebra $\mathfrak{A}$ and a filter F of it, ε^+ is a weak equational implication for $\langle\mathfrak{A}, F\rangle$. □

Next, an *inequality system for* a class K of Σ-algebras is any $\varepsilon \subseteq \mathrm{Eq}^2_\Sigma$ such that the identities of the form

$$\Phi[x_0/((x_k \wedge x_{1-k})[x_0/(x_0 \vee x_0)]), x_1/((x_{2\cdot k} \vee x_{2\cdot(1-k)})[x_0/(x_0 \wedge x_0)])], \tag{4.2}$$

where $\Phi \in \varepsilon$ and $k \in 2$, are true in K.

Remark 4.3. ε^+ is an inequality system for the variety of lattices.[7] □

Remark 4.4. Given a class M of Σ-matrices such that $\mathsf{M} \subseteq \mathsf{PF}(\mathsf{M}{\upharpoonright}\Sigma)$, any equational implication for M is an inequality system for $\mathsf{M}{\upharpoonright}\Sigma$. □

4.1. Associated restricted disjunctive systems.

Lemma 4.5. *Let $\Sigma \supseteq \Sigma^+$, $\mathcal{A}$ a Σ-matrix, Υ an equality determinant for $\mathcal{A}$ and ε both a weak equational implication for $\mathcal{A}$ and an inequality system for $\mathfrak{A}$. Suppose $D^{\mathcal{A}}$ is a prime filter of $\mathfrak{A}$. Then, $\mho^\varepsilon_\Upsilon \triangleq (\bigcup\{\varepsilon[x_0/((\gamma(x_0) \vee \gamma(x_1)) \wedge (\delta(x_2) \vee \delta(x_3))), x_1/((\gamma(x_0) \wedge \gamma(x_1)) \vee (\delta(x_2) \wedge \delta(x_3)))] \mid \gamma, \delta \in \Upsilon\})$ is a disjunctive system for $\mathfrak{A}$.*

Proof. Consider any $\vec{a} \in A^4$. Put $h \triangleq [x_i/a_i]_{i \in 4}$. Then, the metaimplication from left to right in (3.3) is by the fact that the identities (4.2) are true in $\mathfrak{A}$.

Conversely, assume neither $a_0 = a_1$ nor $a_2 = a_3$. Then, there are some $i, j \in 2$ and some $\gamma, \delta \in \Upsilon$ such that $\{\gamma^{\mathfrak{A}}(a_i), \delta^{\mathfrak{A}}(a_{2+j})\} \subseteq D^{\mathcal{A}}$, while $(\{\gamma^{\mathfrak{A}}(a_{1-i}), \delta^{\mathfrak{A}}(a_{3-j})\} \cap D^{\mathcal{A}}) = \varnothing$, in which case, since $D^{\mathcal{A}}$ is a prime filter of $\mathfrak{A}$, we have $((\gamma^{\mathfrak{A}}(a_0) \vee^{\mathfrak{A}} \gamma^{\mathfrak{A}}(a_1)) \wedge^{\mathfrak{A}} (\delta^{\mathfrak{A}}(a_2) \vee^{\mathfrak{A}} \delta^{\mathfrak{A}}(a_3))) \in D^{\mathcal{A}}$, whereas $((\gamma^{\mathfrak{A}}(a_0) \wedge^{\mathfrak{A}} \gamma^{\mathfrak{A}}(a_1)) \vee^{\mathfrak{A}} (\delta^{\mathfrak{A}}(a_2) \wedge^{\mathfrak{A}} \delta^{\mathfrak{A}}(a_3))) \notin D^{\mathcal{A}}$, and so, by (4.1), we get $\mathfrak{A} \not\models (\bigwedge \mho^\varepsilon_\Upsilon)[h]$, as required. □

In view of Remarks 4.2 and 4.3, Lemma 4.5 with $\varepsilon = \varepsilon^+$ incorporates Lemma 11 of [24]. On the other hand, the restricted congruence Σ-scheme $\mho^\varepsilon_\Upsilon$ contains $|\varepsilon \times \Upsilon^2|$ equations, and so the restricted congruence Σ-scheme $\mho^{\varepsilon^+}_\Upsilon$ contains $|\Upsilon^2|$ equations, while that given by Lemma 11 of [24] contains $|2^2 \times \Upsilon^2|$ ones, so Lemma 4.5 provides not merely direct extension but also an advance of Lemma 11 of [24] by reducing the number of equations four times.

[7] It is this fact that justifies the term "inequality system" accepted here.

Lemma 4.6. *Let $\Sigma \supseteq \Sigma^+$ and $\mathfrak{A}$ a non-one-element Σ-algebra. Suppose $\mathfrak{A}{\restriction}\Sigma^+$ is a distributive lattice. Then,* $\mathrm{PF}(\mathfrak{A}) \neq \varnothing$.

Proof. Take any distinct $a, b \in A$. Then, $c \triangleq (a \vee^{\mathfrak{A}} b) \nleq^{\mathfrak{A}} d \triangleq (a \wedge^{\mathfrak{A}} b)$, in which case, by the Prime Ideal Theorem, there is some prime filter $d \notin F \ni c$ of $\mathfrak{A}$, as required. □

Remarks 4.2, 4.3 and Lemmas 4.5 and 4.6 then yield:

Corollary 4.7. *Let $\Sigma \supseteq \Sigma^+$ and K a class of Σ-algebras with equality determinant Υ. Suppose every member of $\mathsf{K}{\restriction}\Sigma^+$ is a distributive lattice. Then, $\mho^{\varepsilon^+}_{\Upsilon}$ is a disjunctive system for K.*

Finally, combining Corollary 3.12 and Lemma 4.5, we immediately get:

Theorem 4.8. *Let $\Sigma \supseteq \Sigma^+$, $\Upsilon \subseteq \mathrm{Tm}^1_\Sigma$, M a [finite]class of [finite]Σ-matrices with weak equational implication ε, $\mathsf{K} \triangleq (\mathsf{M}{\restriction}\Sigma)$, $\mathsf{Q} \triangleq \mathbf{QV}(\mathsf{K})[= \mathbf{PV}(\mathsf{K})]$ and $K \subseteq \infty$. Suppose either both Υ and ε are finite or Q is locally finite (in particular, both M and all members of it are finite), $\{0,2\} \subseteq K$, ε is an inequality system for K and, for each $\mathcal{A} \in \mathsf{M}$, $D^{\mathcal{A}}$ is a prime filter of $\mathfrak{A}$ and Υ is an equality determinant for $\mathcal{A}$. Then, Q is $\mho^{\varepsilon}_{\Upsilon}$-disjunctive, while $\mathrm{SI}^K_{\mathsf{Q}}(\mathsf{Q}) \subseteq [= \mathbf{IS}_{>1}\mathsf{K} =]\mathbf{IS}_{>1}\mathbf{P}^{\mathrm{U}}\mathsf{K} = \mathrm{SI}^{\omega}_{\mathsf{Q}}(\mathsf{Q})$.*

4.2. **Associated restricted implicative systems.** Given any $\Delta \subseteq \mathrm{Tm}^1_\Sigma$, any Σ-matrix $\mathcal{A}$ and any $a \in A$, put $\Delta^{\mathcal{A}}_a \triangleq \{\delta \in \Delta \mid \delta^{\mathfrak{A}}(a) \in D^{\mathcal{A}}\}$.

Lemma 4.9. *Let $\Sigma \supseteq \Sigma^+$, M a class of Σ-matrices, $\Upsilon[\ni x_0]$ an equality determinant for M, ε an equational implication for M, $\bar\varphi \in (\mathrm{Tm}^1_\Sigma)^*$, $\Omega \triangleq \mathrm{img}\,\bar\varphi$ and $\Xi \subseteq \wp(\Omega)$. Suppose $\Xi(\mathsf{M}) \triangleq \{\Omega^{\mathcal{B}}_b \mid b \in B, \mathcal{B} \in \mathsf{M}\} \subseteq \Xi$ and $\mathsf{M} \subseteq \mathsf{PF}(\mathsf{M}{\restriction}\Sigma)$. Then, Ω is an equality determinant for every member of M [if and]only if*

$$\mho^{\varepsilon,\Upsilon}_{\bar\varphi,\Xi} \triangleq (\bigcup\{\varepsilon[x_0/(\wedge\langle(\bar\varphi \cap \Delta) * ((\bar\varphi \cap \Delta)[x_0/x_1]), \upsilon(x_2) \vee \upsilon(x_3)\rangle),$$
$$x_1/(\vee\langle(\bar\varphi \setminus \Delta) * ((\bar\varphi \setminus \Delta)[x_0/x_1]), \upsilon(x_2) \wedge \upsilon(x_3)\rangle)] \mid \Delta \in \Xi, \upsilon \in \Upsilon\})$$

is an implicative system for $\mathsf{M}{\restriction}\Sigma$.

Proof. First, assume Ω is an equality determinant for every member of M. Consider any $\mathfrak{A} \in (\mathsf{M}{\restriction}\Sigma)$ and any $\vec a \in A^4$. Put $h \triangleq [x_k/a_k]_{k \in 4}$. The fact that $(a_2 = a_3) \Rightarrow (\mathfrak{A} \models (\bigwedge \mho^{\varepsilon,\Upsilon}_{\bar\varphi,\Xi})[h])$ is by Remark 4.4 and (4.2) with $k = 1$.

Now, assume $\mathfrak{A} \not\models (\bigwedge \mho^{\varepsilon,\Upsilon}_{\bar\varphi,\Xi})[h]$, in which case there are some $\Delta \in \Xi$ and some $\upsilon \in \Upsilon$ such that

$$\mathfrak{A} \models ((\bigwedge \varepsilon)[x_0/(\wedge\langle(\bar\varphi \cap \Delta) * ((\bar\varphi \cap \Delta)[x_0/x_1]), \upsilon(x_2) \vee \upsilon(x_3)\rangle),$$
$$x_1/(\vee\langle(\bar\varphi \setminus \Delta) * ((\bar\varphi \setminus \Delta)[x_0/x_1]), \upsilon(x_2) \wedge \upsilon(x_3)\rangle)])[h]. \tag{4.3}$$

does not hold, and so, by (4.1), there is some $\mathcal{B} \in (\mathsf{M}{\restriction}\{\mathfrak{A}\}) \subseteq \mathsf{PF}(\{\mathfrak{A}\})$ such that

$$(\wedge\langle(\bar\varphi \cap \Delta) * ((\bar\varphi \cap \Delta)[x_0/x_1]), \upsilon(x_2) \vee \upsilon(x_3)\rangle)^{\mathfrak{A}}[h] \in D^{\mathcal{B}} \not\ni$$
$$(\vee\langle(\bar\varphi \setminus \Delta) * ((\bar\varphi \setminus \Delta)[x_0/x_1]), \upsilon(x_2) \wedge \upsilon(x_3)\rangle)^{\mathfrak{A}}[h]. \tag{4.4}$$

Therefore, since $D^{\mathcal{B}} \in \mathrm{PF}(\mathfrak{A})$, for each $i \in 2$ and every $\delta \in \Omega$, we have $(\delta \in \Delta) \Leftrightarrow (\delta^{\mathfrak{A}}(a_i) \in D^{\mathcal{B}})$. Hence, we get $(\delta^{\mathfrak{A}}(a_0) \in D^{\mathcal{B}}) \Leftrightarrow (\delta^{\mathfrak{A}}(a_1) \in D^{\mathcal{B}})$, for every $\delta \in \Omega$, and so $a_0 = a_1$, because Ω is an equality determinant for $\mathcal{B} \in \mathsf{M}$. Thus, the metaimplication from left to right in (3.3) holds.

Conversely, assume $a_0 = a_1$ and $\mathfrak{A} \models (\bigwedge \mho^{\varepsilon,\Upsilon}_{\bar\varphi,\Xi})[h]$. Consider any $\mathcal{B} \in (\mathsf{M}{\restriction}\{\mathfrak{A}\}) \subseteq \mathsf{PF}(\{\mathfrak{A}\})$ and any $\upsilon \in \Upsilon$. Let us prove, by contradiction, that $((\upsilon^{\mathfrak{A}}(a_2) \vee^{\mathfrak{A}} \upsilon^{\mathfrak{A}}(a_3)) \in D^{\mathcal{B}}) \Rightarrow ((\upsilon^{\mathfrak{A}}(a_2) \wedge^{\mathfrak{A}} \upsilon^{\mathfrak{A}}(a_3)) \in D^{\mathcal{B}})$. For suppose $(\upsilon^{\mathfrak{A}}(a_2) \vee^{\mathfrak{A}} \upsilon^{\mathfrak{A}}(a_3)) \in D^{\mathcal{B}} \not\ni (\upsilon^{\mathfrak{A}}(a_2) \wedge^{\mathfrak{A}} \upsilon^{\mathfrak{A}}(a_3))$. Put $b \triangleq a_0 = a_1 \in A$, and $\Delta \triangleq \Omega^{\mathcal{B}}_b \in \Xi(\mathsf{M}) \subseteq \Xi$. Then, both (4.3), by the above assumption, and (4.4), by the fact that $D^{\mathcal{B}} \in \mathrm{PF}(\mathfrak{A})$, clearly hold. This contradicts to (4.1). In this way, $((\upsilon^{\mathfrak{A}}(a_2) \vee^{\mathfrak{A}} \upsilon^{\mathfrak{A}}(a_3)) \in X) \Rightarrow ((\upsilon^{\mathfrak{A}}(a_2) \wedge^{\mathfrak{A}} \upsilon^{\mathfrak{A}}(a_3)) \in X)$, in which case $(\upsilon^{\mathfrak{A}}(a_2) \in D^{\mathcal{B}}) \Leftrightarrow (\upsilon^{\mathfrak{A}}(a_3) \in D^{\mathcal{B}})$, for $D^{\mathcal{B}} \in \mathrm{PF}(\mathfrak{A})$, and so $a_2 = a_3$, for Υ is an equality determinant for M.

Thus, (3.3) holds, and so $\mathfrak{A}$ is $\mho^{\varepsilon,\Upsilon}_{\bar\varphi,\Xi}$-implicative.[Conversely, assume $x_0 \in \Upsilon$ and $\mho^{\varepsilon,\Upsilon}_{\bar\varphi,\Xi}$ is an implicative system for $\mathsf{M}{\restriction}\Sigma$. Consider any $\mathcal{A} \in \mathsf{M}$, in which case $D^{\mathcal{A}} \in \mathrm{PF}(\mathfrak{A})$, and any $\vec a \in A^2$ such that $(\delta^{\mathfrak{A}}(a_0) \in D^{\mathcal{A}}) \Leftrightarrow (\delta^{\mathfrak{A}}(a_1) \in D^{\mathcal{A}})$, for every $\delta \in \Omega$. We are going to prove, by contradiction, that $a_0 = a_1$. For suppose $a_0 \neq a_1$. Take any $a_2 \in D^{\mathcal{A}} \neq \varnothing$ and any $a_3 \in (A \setminus D^{\mathcal{A}}) \neq \varnothing$, in which case $(a_2 \vee^{\mathfrak{A}} a_3) \in D^{\mathcal{A}} \not\ni (a_2 \wedge^{\mathfrak{A}} a_3)$, for $D^{\mathcal{A}} \in \mathrm{PF}(\mathfrak{A})$. Put $\mathcal{B} \triangleq \mathcal{A}$, $\upsilon \triangleq x_0 \in \Upsilon$ and $h \triangleq [x_k/a_k]_{k \in 4}$, in which case $\mathfrak{A} \models (\bigwedge \mho^{\varepsilon,\Upsilon}_{\bar\varphi,\Xi})[h]$, for $a_0 \neq a_1$, and $\Delta \triangleq \Omega^{\mathcal{A}}_{a_0} = \Omega^{\mathcal{A}}_{a_1} \in \Xi(\mathsf{M}) \subseteq \Xi$. Then, both (4.3) and (4.4), for $D^{\mathcal{A}} \in \mathrm{PF}(\mathfrak{A})$, clearly hold. This contradicts to (4.1). Thus, $a_0 = a_1$, and so Ω is an equality determinant for $\mathcal{A}$, as required.] □

Notice that the cardinality of $\mho^{\varepsilon,\Upsilon}_{\bar\varphi} \triangleq \mho^{\varepsilon,\Upsilon}_{\bar\varphi,\wp(\Omega)}$ is equal to $|\varepsilon \times \Upsilon \times \wp(\Omega)|$, while that of $\mho^{\varepsilon,\Upsilon}_{\bar\varphi,\mathsf{M}} \triangleq \mho^{\varepsilon,\Upsilon}_{\bar\varphi,\Xi(\mathsf{M})} \subseteq \mho^{\varepsilon,\Upsilon}_{\bar\varphi}$ is equal to $\min(|\varepsilon \times \Upsilon \times (\prod_{\mathcal{A} \in \mathsf{M}} A)|, |\varepsilon \times \Upsilon \times \wp(\Omega)|)$. In some finitely-many-valued cases (cf. Subsection 6.3), the latter, though not being uniform, as opposed to the former, may have the crucial advantage of being of *much* lesser cardinality. From now on, the superscript Υ is normally omitted, whenever $\Upsilon = \{x_0\}$.

By Corollary 3.28 and Lemma 4.9, we first get:

Theorem 4.10. *Let $\Sigma \supseteq \Sigma^+$, M a [finite]class of [finite]Σ-matrices, $\mathsf{K} \triangleq (\mathsf{M}{\restriction}\Sigma)$, $\mathsf{Q} \triangleq \mathbf{QV}(\mathsf{K})[= \mathbf{PV}(\mathsf{K})]$, Υ an equality determinant for M, ε an equational implication for M, $\bar\varphi \in (\mathrm{Tm}^1_\Sigma)^*$, $\Omega \triangleq \mathrm{img}\,\bar\varphi$ and $\Xi \subseteq \wp(\Omega)$. Suppose $\Xi(\mathsf{M}) \subseteq \Xi$, $\mathsf{M} \subseteq \mathsf{PF}(\mathsf{K})$ and Ω is an equality determinant for every member of M. Then, $\mho^{\varepsilon,\Upsilon}_{\bar\varphi,\Xi}$ is an implicative system for K. In particular, Q is restricted $\mho^{\varepsilon,\Upsilon}_{\bar\varphi,\Xi}$-implicative, while $\mathrm{SI}^K_{\mathsf{Q}}(\mathsf{Q}) = \mathrm{Si}_{\mathsf{Q}}(\mathsf{Q}) = \mathbf{IS}_{>1}\mathbf{P}^{\mathrm{U}}\mathsf{K}[= \mathbf{IS}_{>1}\mathsf{K}]$, where $\{0,2\} \subseteq K \subseteq \infty$, whenever either both Υ and ε are finite or Q is locally finite (in particular, both M and all members of it are finite).*

4.2.1. *Distributive lattice expansions with finite equality determinant.* Let $\Sigma \subseteq \Sigma^+$.

Remark 4.11. Given a class K of Σ-algebras such that each member of $\mathsf{K}{\upharpoonright}\Sigma^+$ is a distributive lattice, by the Prime Ideal Theorem, ε^+ is an equational implication for $\mathsf{PF}(\mathsf{K})$, and so $\{x_0\}$ is an equality determinant for it (but not for K, unless it consists of merely no-more-than-two-element algebras; cf. Lemma 4.6). □

By $\mathsf{DL}_\Sigma^{[\Upsilon]}$[, where $\Upsilon \subseteq \mathrm{Tm}_\Sigma^1$], we denote the class of all Σ-algebras $\mathfrak{A}$ such that $\mathfrak{A}{\upharpoonright}\Sigma^+$ is a distributive lattice[and Υ is an equality determinant for $\mathfrak{A}$].

Combining Remark 4.11 and Lemma 4.9 with $\Upsilon = \{x_0\}$, $\varepsilon = \varepsilon^+$, $\Xi = \wp(\Omega)$ and $\mathsf{M} = \mathsf{PF}(\{\mathfrak{A}\})$, where $\mathfrak{A} \in \mathsf{DL}_\Sigma$, we first get:

Corollary 4.12. *Let $\Sigma \supseteq \Sigma^+$, $\bar{\varphi} \in (\mathrm{Tm}_\Sigma^1)^*$ and $\Omega \triangleq \mathrm{img}\,\bar{\varphi}$. Then, $\mathsf{DL}_\Sigma^\Omega$ is the class of all $\mho_{\bar{\varphi}}^{\varepsilon^+}$-implicative members of DL_Σ. In particular, $\mathsf{DL}_\Sigma^\Omega \supseteq \mathbf{V}(\varnothing)$ is a universal first-order model class.*

Next, we have:

Lemma 4.13. *Any matrix with equality determinant is simple.*

Proof. Let $\mathcal{A}$ be a Σ-matrix and Υ an equality determinant for it. Consider any $\theta \in \mathrm{Con}(\mathcal{A})$ and any $\langle a, b\rangle \in \theta$. Then, for each $\varphi \in \Upsilon$, we have $\varphi^{\mathfrak{A}}(a)\ \theta\ \varphi^{\mathfrak{A}}(b)$, in which case we get $(\varphi^{\mathfrak{A}}(a) \in D^{\mathcal{A}}) \Leftrightarrow (\varphi^{\mathfrak{A}}(b) \in D^{\mathcal{A}})$, and so $a = b$, as required. □

Corollary 4.14. *Let $\Sigma \supseteq \Sigma^+$, $\mathfrak{A}$ a Σ-algebra and Υ an equality determinant for $\mathfrak{A}$. Suppose $\mathfrak{A}{\upharpoonright}\Sigma^+$ is a distributive lattice. Then, $\mathrm{Con}(\mathfrak{A}) \subseteq \{A^2, \Delta_A\}$. In particular, $\mathfrak{A}$ is simple, whenever it is not one-element.*

Proof. Consider any $\theta \in (\mathrm{Con}(\mathfrak{A}) \setminus \{A^2\})$, in which case $(\mathfrak{A}/\theta){\upharpoonright}\Sigma^+$ is a distributive lattice. Take any $\langle a, b\rangle \in (A^2 \setminus \theta) \neq \varnothing$ and put $c \triangleq (a \vee^{\mathfrak{A}} b)$ and $d \triangleq (a \wedge^{\mathfrak{A}} b)$, in which case $\nu_\theta(c) \nleq^{\mathfrak{A}/\theta} \nu_\theta(d)$, and so, by the Prime Ideal Theorem, there is a prime filter $\nu_\theta(d) \notin F \ni \nu_\theta(c)$ of $\mathfrak{A}/\theta$. Then, $d \notin G \triangleq \nu_\theta^{-1}[F] \ni c$ is a prime filter of $\mathfrak{A}$ such that $\theta[G] \subseteq G$, in which case Υ/θ is an equality determinant for/a congruence of $\langle \mathfrak{A}, G\rangle$, and so, by Lemma 4.13, $\theta = \Delta_A$, as required. □

After all, combining [17] (more specifically, the congruence-distributivity of lattice expansions), Corollaries 4.7, 4.12, 4.14 with the "[]"-optional case of Lemma 3.40 with $\mathsf{S} = (\mathsf{DL}_\Sigma^\Omega \setminus \mathbf{V}(\varnothing))$ and $\mho^{\vee/\supset} = \mho_{\Omega/\bar{\varphi}}^{\varepsilon^+}$, we eventually get:

Theorem 4.15. *Let $\Sigma \supseteq \Sigma^+$, $\bar{\varphi} \in (\mathrm{Tm}_\Sigma^1)^*$, $\Omega \triangleq \mathrm{img}\,\bar{\varphi}$, $\mathsf{K} \subseteq \mathsf{DL}_\Sigma^\Omega$ and $K \subseteq \infty$. Suppose $\{0, 2\} \subseteq K$[and both K and all members of it are finite]. Then, $\mathsf{V} \triangleq \mathbf{QV}(\mathsf{K})$ is a restricted finitely $\mho_{\Omega/\bar{\varphi}}^{\varepsilon^+}$-disjunctive/-implicative variety, while $(\mathsf{V} \cap (\mathsf{DL}_\Sigma^\Omega \setminus \mathbf{V}(\varnothing))) = \mathrm{SI}^K(\mathsf{V}) = \mathrm{Si}(\mathsf{V}) = \mathbf{IS}_{>1}\mathbf{P}^{\mathrm{U}}\mathsf{K}[= \mathbf{IS}_{>1}\mathsf{K}]$ is the class of all non-one-element $\mho_{\Omega/\bar{\varphi}}^{\varepsilon^+}$-disjunctive/-implicative members of V.*

In this way, members of $\mathsf{DL}_\Sigma^\Omega$, where $\Omega \in \wp_\omega(\mathrm{Tm}_\Sigma^1)$, do behave very much like algebras with [dual]discriminator do so (cf. Remark 3.41).

5. Implicativity versus filtrality

Lemma 5.1. *Let Q be a [quasi]variety, I a set, $\overline{\mathfrak{B}} \in \mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q})^I$, $\mathfrak{A}$(a subalgebra of) the direct product of $\overline{\mathfrak{B}}$ and $\theta \in (\mathrm{Con}(\mathfrak{A}) \setminus \{A^2\})$. Suppose $\mathfrak{A}$ is [Q-]congruence-distributive and $\mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q})$ is closed under $\mathbf{P}^{\mathrm{U}}$(and $\mathbf{S}_{>1}$). Then, the following are equivalent:*

(i) *θ is ultra-filtral;*
(ii) *$\theta \in \max(\mathrm{Con}_{[\mathsf{Q}]}(\mathfrak{A}) \setminus \{A^2\})$.*

In particular, each element of $\mathrm{Con}_{[\mathsf{Q}]}(\mathfrak{A})$ is filtral, whenever $\mathrm{Con}_{[\mathsf{Q}]}(\mathfrak{A})$ is co-atomic.

Proof. First, assume (i) holds. Put $\mathfrak{D} \triangleq \prod_{i \in I} \mathfrak{B}_i$. Then, there is some ultra-filter $\mathcal{U}$ on I such that $\theta = \theta_{\mathcal{U}}^A$, in which case $\eta \triangleq \theta_{\mathcal{U}}^D \in \mathrm{Con}(\mathfrak{D})$, while $(\mathfrak{D}/\eta) \in \mathbf{P}^{\mathrm{U}}\,\mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q}) \subseteq \mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q})$.(Moreover, $h \triangleq (\nu_\eta \circ \Delta_A) \in \hom(\mathfrak{A}, \mathfrak{D}/\eta)$, while $(\ker h) = \theta$. Hence, as $\theta \neq A^2$, by (2.1) and Corollary 2.10, we have $(\mathfrak{A}/\theta) \in \mathbf{IS}_{>1}\,\mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q}) \subseteq \mathbf{I}\,\mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q}) \subseteq \mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q})$.) Thus, $(\mathfrak{A}/\theta) \in \mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q})$, and so Lemma 2.8 yields (ii).

Conversely, assume (ii) holds, in which case $A^2 \neq \theta \in \mathrm{Con}_{[\mathsf{Q}]}(\mathfrak{A})$. Then, by Lemmas 2.8, 2.14 with $K = \infty$ and the inclusion $\mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q}) \subseteq \mathrm{SI}_{[\mathsf{Q}]}(\mathsf{Q})$, there is an ultra-filtlar $\vartheta \in \mathrm{Con}(\mathfrak{A})$ such that $\vartheta \subseteq \theta$, in which case $\vartheta \neq A^2$, for $\theta \neq A^2$, and so, by the already-proved metaimplication (i)⇒(ii), we have $\vartheta \in \max(\mathrm{Con}_{[\mathsf{Q}]}(\mathfrak{A}) \setminus \{A^2\})$. Hence, $\theta = \vartheta$. Thus, (i) holds.

Finally, (ii)⇒(i) and the fact that the set of all filters on I is a closure system over $\wp(I)$ complete the argument. □

Theorem 5.2. *Let Q be a [quasi]variety. Then, the following are equivalent:*

(i) *Q is restricted implicative;*
(ii) *Q is [relatively]both congruence-distributive and semi-simple, while the class $\mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q}) \cup \mathbf{V}(\varnothing)$ is a universal first-order model class;*
(iii) *Q is [relatively]both congruence-distributive and semi-simple, while the class $\mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q})$ is closed under both $\mathbf{P}^{\mathrm{U}}$ and $\mathbf{S}_{>1}$;*
(iv) *Q is [relatively]filtral;*
(v) *Q is [relatively]subdirectly filtral.*

Proof. First, (i)⇒(ii) is by Theorems 3.25, 3.26, 3.27 and the fact that quasivarieties are universal first-order model classes. Next, (ii)⇒(iii) is immediate by the fact that universal/first-order model classes are hereditary/ultra-closed (cf., e.g., [13]), while [Q-]simple algebras are not one-element, whereas the class of all non-one-element Σ-algebras is axiomatized by the first-order sentence $\Phi_{>1}$. Further, (iii)⇒(iv) is by Lemma 5.1. Moreover, (v) is a particular case of (iv).

Finally, assume (v) holds, in which case Q is [relatively]semi-simple. Put $\mathsf{K} \triangleq \mathrm{SI}_{[\mathsf{Q}]}(\mathsf{Q})$, $\mathfrak{T} \triangleq \mathfrak{Tm}^4_\Sigma$ and $I \triangleq \{\vartheta \in \mathrm{Con}_{\mathsf{K}}(\mathfrak{T}) \mid (\langle x_0, x_1\rangle \in \vartheta) \Rightarrow (\langle x_2, x_3\rangle \in \vartheta)\}$. Then, for each $i \in I$, $\nu_i \in \hom(\mathfrak{T}, \mathfrak{T}/i)$, in which case $h : T \to (\prod_{i\in I}(T/i)), a \mapsto \langle [a]_i\rangle_{i\in I}$ is a homomorphism from $\mathfrak{T}$ to $\prod_{i\in I}(\mathfrak{T}/i)$, and so onto its subalgebra $\mathfrak{B} \triangleq ((\prod_{i\in I}(\mathfrak{T}/i))\restriction(\mathrm{img}\, h))$. Moreover, for each $i \in I$, $(\pi_i \circ h) = \nu_i$, in which case $\pi_i[B] = \pi_i[h[T]] = (T/i)$, and so $\mathfrak{B}$ is a subdirect product of $\langle \mathfrak{T}/i\rangle_{i\in I} \in \mathsf{K}^I$. Then, by (v), there is a filter $\mathcal{F}$ on I such that $\theta \triangleq \mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{Q}]}(\langle h(x_0), h(x_1)\rangle) = \theta^B_{\mathcal{F}}$. Moreover, $\langle h(x_0), h(x_1)\rangle \in \theta$, and so $E(h(x_0), h(x_1)) \in \mathcal{F}$. On the other hand, $E(h(x_0), h(x_1)) \subseteq E(h(x_2), h(x_3))$, in which case $E(h(x_2), h(x_3)) \in \mathcal{F}$, and so $\langle h(x_2), h(x_3)\rangle \in \theta$. Then, by the following claim, $\mho \triangleq (\ker h)$ is a restricted congruence Σ-scheme for Q such that $\mathfrak{B} \models (\bigwedge \mho)[h\restriction V_4]$:

Claim 5.3. *Let $\mathfrak{A}$ be a Σ-algebra, $\alpha \in (\infty\setminus 4)$, $h \in \hom(\mathfrak{Tm}^\alpha_\Sigma, \mathfrak{A})$ and P a prevariety of Σ-algebras. Suppose $(\mathrm{img}\, h) = A$ and $\langle h(x_2), h(x_3)\rangle \in \mathrm{Cg}^{\mathfrak{A}}_{\mathsf{P}}(\langle h(x_0), h(x_1)\rangle)$. Then, $\mho \triangleq (\ker h)$ is a congruence Σ-scheme of rank α for P such that the following holds:*

$$\mathfrak{A} \models (\exists_{\alpha\setminus 4} \bigwedge \mho)[h\restriction V_4]. \tag{5.1}$$

Proof. First, (5.1) is by the inclusion $\mho \subseteq \ker h$. Finally, consider any $\mathfrak{B} \in \mathsf{P}$ and any $g \in \hom(\mathfrak{Tm}^\alpha_\Sigma, \mathfrak{B})$ such that $(\{x_0 \approx x_1\} \cup \mho) \subseteq \ker g$. Then, by Lemma 2.8, we have $(\ker g) = g^{-1}[\Delta_B] \in \mathrm{Con}_{\mathsf{P}}(\mathfrak{Tm}^\alpha_\Sigma)$, for $\Delta_B \in \mathrm{Con}_{\mathsf{P}}(\mathfrak{B})$, so, by Corollary 2.11, we eventually get $(\ker g) \supseteq \mathrm{Cg}^{\mathfrak{Tm}^\alpha_\Sigma}_{\mathsf{P}}(\{x_0 \approx x_1\} \cup (\ker h)) = h^{-1}[\mathrm{Cg}^{\mathfrak{A}}_{\mathsf{P}}(\langle h(x_0), h(x_1)\rangle)] \ni \langle x_2, x_3\rangle$. Thus, (3.5) is true in P, as required. □

In that case:

$$(\mathfrak{T}/i) \models (\bigwedge \mho)[\nu_i\restriction V_4], \quad \text{for all } i \in I. \tag{5.2}$$

We are going to argue that $\mho$ is an implicative system for K. For consider any $\mathfrak{A} \in \mathsf{K} \subseteq \mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q})$ and any $\bar{a} \in A^4$, in which case $e \triangleq [x_i/a_i]_{i\in 4}$ is extended to the equally-denoted homomorphism from $\mathfrak{T}$ to $\mathfrak{A}$. Then, the metaimplication from right to left in (3.3) is by Remark 3.23(i). Conversely, assume $(a_0 = a_1) \Rightarrow (a_2 = a_3)$. In case $\mathrm{img}\, e$ is a singleton, we clearly have $\mathfrak{A} \models (\bigwedge \mho)[e]$. Otherwise, by Corollary 2.21, $\mathfrak{D} \triangleq (\mathfrak{A}\restriction(\mathrm{img}\, e)) \in \mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q}) \subseteq \mathsf{K}$, in which case $e \in \hom(\mathfrak{T}, \mathfrak{D})$ is surjective, and so, by Lemma 2.8, $\vartheta \triangleq (\ker e) = e^{-1}[\Delta_D] \in \mathrm{Con}_{\mathsf{K}}(\mathfrak{T})$, for $\mathsf{K} \ni \mathfrak{D}$ is closed under $\mathbf{I}$, in view of Corollary 2.10. Hence, $\vartheta \in I$. Moreover, by the Homomorphism Theorem, $(e \circ \nu_\vartheta^{-1})$ is an isomorphism from $\mathfrak{T}/\vartheta$ onto $\mathfrak{D}$. Therefore, by (5.2) with $i = \vartheta$, we eventually get $\mathfrak{D} \models (\bigwedge \mho)[e]$, in which case $\mathfrak{A} \models (\bigwedge \mho)[e]$, and so (i) holds, as required. □

Combining [4] with Theorem 5.2, we first get:

Corollary 5.4. *Let V be a variety. Then, the following are equivalent:*

(i) V *is discriminator;*
(ii) V *is both restricted implicative and congruence-permutable;*
(iii) V *is arithmetical and semi-simple, while* $\mathrm{Si}(\mathsf{V}) \cup \mathbf{V}(\varnothing)$ *is a universal first-order model class;*
(iv) V *is arithmetical and semi-simple, while* $\mathrm{Si}(\mathsf{V})$ *is closed under both* $\mathbf{P}^{\mathrm{U}}$ *and* $\mathbf{S}_{>1}$.

Corollary 5.4 characterizes discriminator varieties in terms of their most well-known and evident properties, in view of Remarks 3.38 and 3.41. And what is more, it definitely shows that Corollary 4.12 of [26] cannot be essentially strengthened by replacing "discriminator" with "congruence-permutable equational". On the other hand, since [dual] discriminator quasivarieties are varieties, Corollary 5.4 may hardly have any meaningful quasi-equational relativization.

Corollary 5.5. *Let Q be a quasivariety of Σ-algebras. Then, Q is restricted implicative iff it is relatively congruence-distributive and generated by some $\mathsf{K} \subseteq \mathsf{Q}$ such that $\mathbf{S}_{>1}\mathbf{P}^{\mathrm{U}}\mathsf{K} \subseteq \mathrm{Si}_{\mathsf{Q}}(\mathsf{Q})$, in which case:*

(i) $\mathrm{Si}_{\mathsf{Q}}(\mathsf{Q}) = \mathbf{IS}_{>1}\mathbf{P}^{\mathrm{U}}\mathsf{K} = \mathrm{SI}^K_{\mathsf{Q}}(\mathsf{Q})$*, for all* $\{0, 2\} \subseteq K \subseteq \infty$*;*
(ii) Q *is a variety iff it is congruence-distributive and every member of* $\mathbf{S}_{>1}\mathbf{P}^{\mathrm{U}}\mathsf{K}$ *is simple.*

Proof. The "only if" part is by Theorem 5.2(i)⇒(iii), when taking $\mathsf{K} = \mathrm{Si}_{\mathsf{Q}}(\mathsf{Q})$. Conversely, assume Q is relatively congruence-distributive and generated by some $\mathsf{K} \subseteq \mathsf{Q}$ such that $\mathbf{S}_{>1}\mathbf{P}^{\mathrm{U}}\mathsf{K} \subseteq \mathrm{Si}_{\mathsf{Q}}(\mathsf{Q})$. Then, by Corollaries 2.10 and 2.16, we have $\mathrm{Si}_{\mathsf{Q}}(\mathsf{Q}) \subseteq \mathrm{SI}^K_{\mathsf{Q}}(\mathsf{Q}) \subseteq \mathbf{IS}_{>1}\mathbf{P}^{\mathrm{U}}\mathsf{K} \subseteq \mathbf{I}\,\mathrm{Si}_{\mathsf{Q}}(\mathsf{Q}) \subseteq \mathrm{Si}_{\mathsf{Q}}(\mathsf{Q})$, where $\{0, 2\} \subseteq K \subseteq \infty$, in which case we get (i), and so, by it(and Theorem 2.13) with $K = \infty$, we conclude that(Q is relatively semi-simple, while) $\mathrm{Si}_{\mathsf{Q}}(\mathsf{Q})$ is closed under both $\mathbf{S}_{>1}$ and, by Corollary 2.7, $\mathbf{P}^{\mathrm{U}}$, for any member of $\mathrm{Si}_{\mathsf{Q}}(\mathsf{Q})$ is non-one-element, while the class of all non-one-elment Σ-algebras is axiomatized by $\Phi_{>1}$, whereas first-order model classes are closed under $\mathbf{P}^{\mathrm{U}}$ (cf., e.g., [13]). In this way, (i) with $K = \infty$, Theorems 2.13 with $K = \infty$, 5.2(iii)⇒(i), 3.31 and Corollary 2.10 complete the argument. □

Combining [17] (more specfically, the congruence-distributivity of lattice expansions) with Corollaries 2.10, 5.5 and the "[]"-option-free case of Lemma 3.40 with $\mathsf{S} = \mathrm{Si}(\mathsf{Q})$, we then get:

Corollary 5.6. *Let K be a class of Σ-algebras, $K \subseteq \infty$ and $\mathsf{Q} \triangleq \mathbf{QV}(\mathsf{K})$. Suppose $\Sigma^+ \subseteq \Sigma$, each member of $\mathsf{K}\restriction\Sigma^+$ is a lattice, every member of $\mathbf{S}_{>1}\mathbf{P}^{\mathrm{U}}\mathsf{K}$ is simple and $\{0, 2\} \subseteq K$. Then, the following hold:*

(i) Q *is a restricted implicative variety;*
(ii) $\mathrm{Si}(\mathsf{Q}) = \mathbf{IS}_{>1}\mathbf{P}^{\mathrm{U}}\mathsf{K} = \mathrm{SI}^K(\mathsf{Q})$.

5.1. **Restricted implicativity versus local finiteness.**

Lemma 5.7. *Let* Q *be a locally-finite [quasi]variety and* $\mathfrak{A} \in \mathsf{Q}$*. Suppose* $|A| > 1$ *and each finite non-one-element subalgebra of* $\mathfrak{A}$ *is ([*Q*-])simple. Then, so is* $\mathfrak{A}$*.*

Proof. Consider any $\theta \in (\mathrm{Con}_{([\mathsf{Q}])}(\mathfrak{A}) \setminus \{\Delta_A\})$ and any $c, d \in A$. Take any $\langle a, b\rangle \in (\theta \setminus \Delta_A) \neq \varnothing$. Let $\mathfrak{B}$ be the subalgebra of $\mathfrak{A}$ generated by $\{a, b, c, d\}$. Then, $\mathfrak{B} \in \mathsf{Q}$ is finitely generated, and so finite. Moreover, as $a \neq b$, $\mathfrak{B}$ is not one-element, in which case it is ([Q-])simple. On the other hand, by Lemma 2.8 with $h = \Delta_B$, we have $\Delta_B \not\ni \langle a, b\rangle \in (\theta \cap B^2) = h^{-1}[\theta] \in \mathrm{Con}_{([\mathsf{Q}])}(\mathfrak{B})$, in which case $(\theta \cap B^2) = B^2$, and so $\langle c, d\rangle \in B^2 \subseteq \theta$. Thus, $\theta = A^2$, as required. $\square$

Since each member of $\mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q})$ is not-one-element, i.e., satisfies $\Phi_{>1}$, and so is any ultraproduct of them, by Corollaries 2.10, 2.19 with $\mathsf{K} = \mathsf{S} = \mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q})$ and Lemma 5.7, we immediately get:

Corollary 5.8. *Let* Q *be a locally-finite [quasi]variety. Then,* $\mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q})$ *is closed under both* $\mathbf{P}^{\mathrm{U}}$ *and* $\mathbf{S}_{>1}$ *iff it is closed under* $\mathbf{S}_{>1}$ *iff* $(\mathbf{S}_{>1}\,\mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q}))_{<\omega} \subseteq \mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q})$*.*

Then, combining Corollary 5.8 with Theorem 5.2 and Corollary 5.4, we get, respectively:

Corollary 5.9. *Let* Q *be a locally-finite [quasi]variety. Then, the following are equivalent:*

(i) Q *is restricted implicative;*
(ii) Q *is [relatively]both congruence-distributive and semi-simple, while the class* $\mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q})$ *is closed under* $\mathbf{S}_{>1}$*;*
(iii) Q *is [relatively]both congruence-distributive and semi-simple, while it holds that* $(\mathbf{S}_{>1}\,\mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q}))_{<\omega} \subseteq \mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q})$*.*

Corollary 5.10. *Let* V *be a locally-finite variety. Then, the following are equivalent:*

(i) V *is discriminator;*
(ii) V *is both arithmetical and semi-simple, while* $\mathrm{Si}(\mathsf{V})$ *is closed under* $\mathbf{S}_{>1}$*;*
(iii) V *is both arithmetical and semi-simple, while* $(\mathbf{S}_{>1}\,\mathrm{Si}(\mathsf{V}))_{<\omega} \subseteq \mathrm{Si}(\mathsf{V})$*.*

Likewise, combining Corollaries 2.10, 5.5, 2.19 with $\mathsf{S} = \mathrm{Si}_{([\mathsf{Q}])}(\mathsf{Q})$ and Lemma 5.7, we immediately get:

Corollary 5.11. *Let* K *be a [finite]class of[finite]* Σ*-algebras and* $K \subseteq \infty$*. Suppose* $\mathsf{Q} \triangleq \mathbf{QV}(\mathsf{K})[= \mathbf{PV}(\mathsf{K})]$ *is both locally finite (in particular, both* K *and all members of it are finite) and relatively congruence-distributive,* $(\mathbf{S}_{>1})_{<\omega}\mathsf{K} \subseteq \mathrm{Si}_{\mathsf{Q}}(\mathsf{Q})$ *and* $\{0, 2\} \subseteq K$*. Then, the following hold:*

(i) Q *is restricted implicative;*
(ii) $\mathrm{Si}_{\mathsf{Q}}(\mathsf{Q}) = \mathbf{IS}_{>1}\mathbf{P}^{\mathrm{U}}\mathsf{K}[= \mathbf{IS}_{>1}\mathsf{K}] = \mathrm{SI}^{K}_{\mathsf{Q}}(\mathsf{Q})$*;*
(iii) Q *is a variety iff it is congruence-distributive, whereas every member of* $(\mathbf{S}_{>1}\mathsf{K})_{<\omega}$ *is simple.*

In this way, combining the "[]"-option-free case of Lemma 3.40, Corollaries 2.10, 5.11, 2.19 with $\mathsf{S} = \mathrm{Si}(\mathsf{Q})$ and with [17] (more specifically, the congruence-distributivity of lattice expansions), we get the following valuable generic result covering, in particular, expansions of both distributive and De Morgan lattices (cf. Subsections 6.1 and 6.2, respectively):

Corollary 5.12. *Let* K *be a [finite]class of[finite]* Σ*-algebras and* $K \subseteq \infty$*. Suppose* $\mathsf{Q} \triangleq \mathbf{QV}(\mathsf{K})[= \mathbf{PV}(\mathsf{K})]$ *is locally finite (in particular, both* K *and all members of it are finite),* $\Sigma^{+} \subseteq \Sigma$*, each member of* $\mathsf{K}{\restriction}\Sigma^{+}$ *is a lattice, every member of* $(\mathbf{S}_{>1}\mathsf{K})_{<\omega}$ *is simple and* $\{0, 2\} \subseteq K$*. Then, the following hold:*

(i) Q *is a restricted implicative variety;*
(ii) $\mathrm{Si}(\mathsf{Q}) = \mathbf{IS}_{>1}\mathbf{P}^{\mathrm{U}}\mathsf{K}[= \mathbf{IS}_{>1}\mathsf{K}] = \mathrm{SI}^{K}(\mathsf{Q})$*.*

5.1.1. *Implicativity versus disjunctivity.*

Theorem 5.13. *A locally-finite [quasi]variety* Q *is restricted implicative iff it is both (finitely)restricted disjunctive and [relatively]semi-simple.*

Proof. First, the "only if" part is by Remark 3.23(v) and Theorem 3.25. Conversely, assume Q is both restricted disjunctive and [relatively]semi-simple. Then, by Theorem 3.10, Q is [relatively]congruence-distributive. Put $\mathsf{K} \triangleq \mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q}) \subseteq \mathsf{S} \triangleq \mathrm{SI}^{\omega}_{[\mathsf{Q}]}(\mathsf{Q})$. Consider any finite $\mathfrak{B} \in \mathbf{S}_{>1}\mathsf{K} \subseteq \mathbf{S}_{>1}\mathsf{S}$. Then, by Proposition 3.11, we conclude that $\mathfrak{B} \in \mathsf{S}$, in which case it, being finite, is [Q-]subdirectly-irreducible, and so [Q-]simple. In this way, Corollary 5.9(iii)⇒(i) completes the argument. $\square$

This provides (though non-constructively) the relationship between restricted implicativity and restricted disjunctivity inverse to that given by Remark 2.4 of [26] and looks especially non-trivial, because implication is not definable via disjunction alone in the classical logic. And what is more, combining Theorem 5.13 with Corollary 5.4, we get a one more characterization of locally finite discriminator varieties in terms of their well-known and evident properties (in view of Remarks 3.38 and 3.41):

Corollary 5.14. *A locally finite variety is discriminator iff it is (finitely)disjunctive, semi-simple and congruence-permutable.*

5.2. **Parameterized implicativity versus direct filtrality.**

Lemma 5.15. *Let* Q *be a [quasi]variety of* Σ*-algebras,* $\mathsf{K} \triangleq \mathrm{SI}_{[\mathsf{Q}]}(\mathsf{Q})$*,* I *a set and* $\overline{\mathfrak{A}} \in \mathsf{K}^{I}$*. Suppose* Q *is implicative. Then, any (finitely-generated)[*Q*-]congruence of* $\mathfrak{B} \triangleq (\prod_{i\in I} \mathfrak{A}_i)$ *is (principally)filtral.*

Proof. Let $\mho \in \wp(\mathrm{Eq}_\Sigma^\alpha)$, where $\alpha \in (\infty \setminus 4)$, be an implicative system for K.

(First of all, by induction on the cardinality of any $X \in \wp_\omega(B^2)$, we are going to argue that $\eta \triangleq \mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{Q}]}(X)$ is principally filtral. In case $X = \varnothing$, we clearly have $\eta = \Delta_B = \theta^B_{\{I\}}$, while $\{I\} = \wp(I, I)$ is a principal filter on I.

Otherwise, take any $\langle \bar{a}, \bar{b}\rangle \in X \neq \varnothing$. Put $Y \triangleq (X \setminus \{\langle \bar{a}, \bar{b}\rangle\}) \in \wp_\omega(B^2)$, in which case $|Y| < |X|$, and so, by induction hypothesis, $\theta \triangleq \mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{Q}]}(Y)$ is principally filtral. Then, there is some $J \subseteq I$ such that $\theta = \theta^B_{\wp(J,I)} = (B^2 \cap \bigcap_{j\in J} \ker \pi_j)$, in which case $h : (\prod_{i\in I} A_i) \to (\prod_{j\in J} A_j), \bar{e} \mapsto \langle e_j\rangle_{j\in J}$ is a surjective homomorphism from $\mathfrak{B}$ onto $\mathfrak{D} \triangleq (\prod_{j\in J} \mathfrak{A}_j) \in \mathsf{Q}$ such that

$$(\pi_j \circ h) = (\pi_j{\restriction}B), \tag{5.3}$$

for all $j \in J$, and so $(\ker h) = \theta$. Consider any $\bar{c}, \bar{d} \in D$. First, assume $\langle \bar{c}, \bar{d}\rangle \in \vartheta \triangleq \mathrm{Cg}^{\mathfrak{D}}_{[\mathsf{Q}]}(\langle h(\bar{a}), h(\bar{b})\rangle)$. Then, for every $j \in K \triangleq (J \cap E(\bar{a}, \bar{b}))$, by Corollary 2.11 and (5.3), we have $\langle c_j, d_j\rangle \in \mathrm{Cg}^{\mathfrak{A}_j}_{[\mathsf{Q}]}(\langle a_j, b_j\rangle) = \Delta_{A_j}$, in which case $j \in E(\bar{c}, \bar{d})$, and so $K \subseteq E(\bar{c}, \bar{d})$. Conversely, assume $K \subseteq E(\bar{c}, \bar{d})$. Then, for every $j \in J$, we have $(a_j = b_j) \Rightarrow (c_j = d_j)$, in which case, by (3.3), we get $\mathfrak{A}_j \models (\exists_{\alpha\setminus 4} \bigwedge \mho)[x_0/a_j, x_1/b_j, x_2/c_j, x_3/d_j]$, and so, by (5.3), we get $\mathfrak{D} \models (\exists_{\alpha\setminus 4} \bigwedge \mho)[x_0/h(\bar{a}), x_1/h(\bar{b}), x_2/\bar{c}, x_3/\bar{d}]$. Therefore, $\mathsf{Q} \ni (\mathfrak{D}/\vartheta) \models (\exists_{\alpha\setminus 4} \bigwedge \mho)[x_0/\nu_\vartheta(h(\bar{a})), x_1/\nu_\vartheta(h(\bar{b})), x_2/\nu_\vartheta(\bar{c}), x_3/\nu_\vartheta(\bar{d})]$. On the other hand, $\langle h(\bar{a}), h(\bar{b})\rangle \in \vartheta$. Hence, by Remark 3.23(i,ii), we conclude that $\langle \bar{c}, \bar{d}\rangle \in \vartheta$. Thus, $\vartheta = \theta^D_{\wp(K,J)}$. In this way, by Corollary 2.11 and (5.3), we eventually get $\eta = \mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{Q}]}((\ker h) \cup \{\langle \bar{a}, \bar{b}\rangle\}) = h^{-1}[\vartheta] = h^{-1}[D^2 \cap \bigcap_{k\in K} \ker \pi_k] = (B^2 \cap \bigcap_{k\in K} h^{-1}[\ker \pi_k]) = (B^2 \cap \bigcap_{k\in K} \ker(\pi_k{\restriction}B)) = (B^2 \cap \bigcap_{k\in K} \ker \pi_k) = \theta^B_{\wp(K,I)}$. This completes the argument by induction.)

Thus, by Lemma 3.5, there is some embedding f of the poset $\mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{Q}]}[\wp_\omega(B^2)]$ into that $\boldsymbol{F}$ of all filters on I, both ones being ordered by inclusion, such that, for every $\theta \in \mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{Q}]}[\wp_\omega(B^2)]$, it holds that $\theta = \theta^B_{f(\theta)}$.

Finally, consider any $\theta \in \mathrm{Con}_{[\mathsf{Q}]}(\mathfrak{B})$. Then, $\Theta \triangleq \mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{Q}]}[\wp_\omega(\theta)] \subseteq \mathrm{Cg}^{\mathfrak{B}}_{[\mathsf{Q}]}[\wp_\omega(B^2)]$ is upward-directed, and so is $f[\Theta] \subseteq \boldsymbol{F}$, in which case $\mathcal{F} \triangleq (\bigcup f[\Theta]) \in \boldsymbol{F}$, for $\boldsymbol{F}$ is inductive. And what is more, we have $\theta^B_{\mathcal{F}} = (\bigcup_{\theta\in\Theta} \theta^B_{f(\theta)}) = (\bigcup \Theta) = \theta$, in view of Corollary 2.12, as required. □

Lemma 5.16. *Any [relatively]directly filtral [quasi]variety* Q *is implicative.*

Proof. By contradiction. For suppose Q is not implicative. Then, for each $\mho \in S \triangleq (\bigcup\{\wp_\omega(\mathrm{Eq}^m_\Sigma) \mid m \in (\omega \setminus 4)\}$, the class $\mathsf{K}_\mho$ of all non-$\mho$-implicative members of $\mathsf{K} \triangleq \mathrm{SI}_{[\mathsf{Q}]}(\mathsf{Q})$ is not empty, in which case $\varnothing \neq O_\mho \triangleq \{|A| \mid \mathfrak{A} \in \mathsf{K}_\mho\} \subseteq \infty$, and so $\alpha_\mho \triangleq (\bigcap O_\mho) \in O_\mho$, in which case $\mathsf{K}'_\mho \triangleq \{\mathfrak{A} \in \mathsf{K}_\mho \mid |A| = \alpha_\mho\} \neq \varnothing$. In this way, $\alpha \triangleq (\bigcup\{\alpha_\mho \mid \mho \in S\}) \in \infty$, for S is a set (viz., is not a proper class).

Put $\mathfrak{T} \triangleq \mathfrak{Tm}^{4+\alpha}_\Sigma$, $I \triangleq \{\vartheta \in \mathrm{Con}_{\mathsf{K}}(\mathfrak{T}) | (\langle x_0, x_1\rangle \in \vartheta) \Rightarrow (\langle x_2, x_3\rangle \in \vartheta)\}$, and, for each $i \in I$, $a_i \triangleq [x_0]_i$, $b_i \triangleq [x_1]_i$, $c_i \triangleq [x_2]_i$, $d_i \triangleq [x_3]_i$ and $\mathfrak{A}_i \triangleq (\mathfrak{T}/i) \in \mathsf{K}$, in which case $(i \in E(\bar{a}, \bar{b})) \Rightarrow (i \in E(\bar{c}, \bar{d}))$, and so $E(\bar{a}, \bar{b}) \subseteq E(\bar{c}, \bar{d})$. Finally, set $\mathfrak{B} \triangleq \prod_{i\in I} \mathfrak{A}_i$, in which case there is a filter $\mathcal{F}$ on I such that $\theta \triangleq \mathrm{Cg}^{\mathfrak{B}}_{\mathsf{Q}}(\langle \bar{a}, \bar{b}\rangle) = \theta^B_{\mathcal{F}}$. Then, as $\langle \bar{a}, \bar{b}\rangle \in \theta$, we have $E(\bar{a}, \bar{b}) \in \mathcal{F}$, in which case $E(\bar{c}, \bar{d}) \in \mathcal{F}$, and so $\langle \bar{c}, \bar{d}\rangle \in \theta$. Take any bijection $e : B \to \beta \triangleq |B|$. As it is well known (cf., e.g., [14]), for any $\gamma, \delta \in \infty$, $f^\gamma_\delta : \delta \to ((\gamma + \delta) \setminus \gamma), \epsilon \mapsto (\gamma + \epsilon)$ is injective, in which case $g \triangleq (f^4_\beta \circ e) : B \to ((4 + \beta) \setminus 4)$ is injective too, and so $h \triangleq (\{\langle x_\epsilon, g^{-1}(\epsilon)\rangle \mid \epsilon \in (\mathrm{img}\, g)\} \cup \{\langle x_\epsilon, \bar{a}\rangle \mid \epsilon \in (((4 + \beta) \setminus 4) \setminus (\mathrm{img}\, g))\} \cup \{\langle x_j, \langle [x_j]_i\rangle_{i\in I}\rangle\} \mid j \in 4\}) : V_{4+\beta} \to B$ is a surjection to be extended to the equally-denoted surjective homomorphism from $\mathfrak{T}$ onto $\mathfrak{B}$. Hence, by Remarks 3.22, 3.7 and Claim 5.3, there is some finite congruence Σ-scheme $\mho \in S$ of some finite rank $n \in (\omega \setminus 4)$ for Q such that $\mathfrak{B} \models (\exists_{n\setminus 4} \bigwedge \mho)[x_0/\bar{a}, x_1/\bar{b}, x_2/\bar{c}, x_3/\bar{d}]$, in which case we get:

$$\mathfrak{A}_i \models (\exists_{n\setminus 4} \bigwedge \mho)[x_0/a_i, x_1/b_i, x_2/c_i, x_3/d_i], \qquad \text{for all } i \in I. \tag{5.4}$$

Take any $\mathfrak{A} \in \mathsf{K}'_\mho \neq \varnothing$, in which case $\mathfrak{A} \in \mathsf{K}_\mho$, while $|A| = \alpha_\mho \subseteq \alpha$, and so there is an injection $e' : A \to \alpha$. Consider any $\bar{a'} \in A^4$. Then, the metaimplication from right to left in (3.3) is by Remark 3.23(i). Conversely, assume $(a'_0 = a'_1) \Rightarrow (a'_2 = a'_3)$. Then, $g' \triangleq (f^4_\alpha \circ e') : A \to ((4 + \alpha) \setminus 4)$ is injective, in which case $h' \triangleq (\{\langle x_\delta, {g'}^{-1}(\delta)\rangle \mid \delta \in (\mathrm{img}\, g')\} \cup \{\langle x_\delta, a'_0\rangle \mid \delta \in (((4 + \alpha) \setminus 4) \setminus (\mathrm{img}\, g'))\} \cup \{\langle x_i, a'_i\rangle \mid i \in 4\}) : V_{4+\alpha} \to A$ is a surjection to be extended to the equally-denoted surjective homomorphism from $\mathfrak{T}$ onto $\mathfrak{A}$. Then, by Lemma 2.8, $\vartheta \triangleq (\ker h') = {h'}^{-1}[\Delta_A] \in \mathrm{Con}_{\mathsf{K}}(\mathfrak{T})$, for $\mathsf{K} \ni \mathfrak{A}$ is closed under $\mathbf{I}$, in view of Corollary 2.10, in which case $\vartheta \in I$, while, by the Homomorphism Theorem, $h' \circ \nu_\vartheta^{-1}$ is an isomorphism from $\mathfrak{A}_\vartheta$ onto $\mathfrak{A}$, and so, by (5.4) with $i = \vartheta$, we eventually get $\mathfrak{A} \models (\exists_{n\setminus 4} \bigwedge \mho)[x_i/a'_i]_{i\in 4}$. Thus, $\mathfrak{A}$ is $\mho$-implicaive. This contradicts to the fact that $\mathfrak{A} \in \mathsf{K}_\mho$, as required. □

Theorem 5.17. *Let* Q *be a [quasi]variety. Then, the following are equivalent:*

(i) Q *is implicative;*
(ii) Q *is [relatively]directly filtral;*
(iii) Q *is [relatively]both directly congruence-distributive and semi-simple, while* $\mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q})$ *is a first-order model class;*
(iv) Q *is [relatively]both directly congruence-distributive and semi-simple, while* $\mathrm{Si}_{[\mathsf{Q}]}(\mathsf{Q})$ *is closed under* $\mathbf{P}^{\mathrm{U}}$.

Proof. First of all, (i)⇔(ii) is by Lemmas 5.15 and 5.16. Next, (i&ii)⇒(iii) is by Theorems 3.6, 3.25, 3.26 and the fact that any [Q-]simple algebra is not one-element, while the class of all non-one-element similar algebras is axiomatized by the first-order sentence $\Phi_{>1}$, whereas quasivarieties are first-order model classes. Further, (iii)⇒(iv) is by the fact that first-order model classes are closed under $\mathbf{P}^{\mathrm{U}}$ (cf., e.g., [13]). Finally, (iv)⇒(ii) is by Lemma 5.1. □

By Theorems 3.37 and 5.17(i)⇒(iii), we then have:

Corollary 5.18. *Any [relatively]semi-simple [quasi]variety with EDP[R]C is [relatively]directly congruence-distributive.*

6. Examples

To demonstrate all the power of the generic elaboration presented in the previous sections, we repeat certain key well-known results, providing them with transparent concise argumentation just for the expository and methodological purpose. When discussing particular examples, to demonstrate the applicability of Theorems 3.13/5.2, we start from giving non-constructive proofs of their finite restricted disjunctivity/implicativity and only then present constructive ones based upon the conception of equality determinant.

6.1. **Two-valued expansions of distributive lattices.** The variety of distributive lattices, viewed as Σ^+-algebras, is denoted by DL. Fix any $\Sigma \supseteq \Sigma^+$.

Let $\mathfrak{D}_n$, where $n \in (\omega \setminus 2)$, be the distributive lattice given by the chain poset n ordered by inclusion. Consider any expansion $\mathfrak{A}$ of $\mathfrak{D}_2$, that is, a Σ-algebra such that $(\mathfrak{A}{\restriction}\Sigma^+) = \mathfrak{D}_2$. Then, since any two-element algebra is simple and has no proper non-one-element subalgebra, every member of $\mathbf{S}_{>1}\mathfrak{A} = \{\mathfrak{A}\}$ is simple. Hence, by Corollary 5.12, we *immediately* conclude that $\mathsf{QV}(\mathfrak{A})$ is a restricted implicative (and so finitely restricted disjunctive; cf. Remark 3.23(v)) variety, its simple/(finitely-)subdirectly-irreducible members being exactly isomorphic copies of $\mathfrak{A}$.

On the other hand, we have the following well-known result:

Proposition 6.1. $\mathsf{DL} = \mathbf{PV}(\mathfrak{D}_2)$.

Proof. With using Remark 2.4. For consider any $\mathfrak{B} \in \mathsf{DL}$ and any distinct $a, b \in B$. Then, $c \triangleq (a \vee^{\mathfrak{B}} b) \nleq^{\mathfrak{B}} d \triangleq (a \wedge^{\mathfrak{B}} b)$, so, by the Prime Ideal Theorem, there is some prime filter F of $\mathfrak{B}$ such that $d \notin F \ni c$, in which case $(a \in F) \Leftrightarrow (b \notin F)$. Then, by the following immediate observation, $h \triangleq \chi^F_B \in \hom(\mathfrak{B}, \mathfrak{D}_2)$:

Claim 6.2. *Let $\mathfrak{C}$ be a Σ-algebra, $n \in (\omega \setminus 2)$ and $\vec{F} \in \wp(C)^n$. Suppose $F_0 = C$ and, for each $i \in (n \setminus 1)$, $F_i \subseteq F_{i-1}$ is a prime filter of $\mathfrak{C}$. Then, $\chi^{\vec{F}} \in \hom(\mathfrak{C}, \mathfrak{D}_n)$.*

Moreover, $(h(a) = 1) \Leftrightarrow (h(b) \neq 1)$, and so $h(a) \neq h(b)$, as required. □

In this way, Lemma 3.36 does provide a new, generic and quite transparent insight into the issue of REDPC for DL going back to [7].

However, such argumentation is not constructive. Nevertheless, $\{1\}$ is the only prime filter of $\mathfrak{A}$, while $\{x_0\}$ is an equality determinant for $\langle \mathfrak{A}, \{1\} \rangle$, and so for $\mathfrak{A}$. Hence, by Theorem 4.15, we immediately conclude that $\mathbf{QV}(\mathfrak{A})$ (in particular, DL; cf. Proposition 6.1) is restricted finitely $\mho^{\varepsilon^+}_{\{x_0\}/\langle x_0 \rangle}$-disjunctive/-implicative, the disjunctive/implicative system $\mho^{\varepsilon^+}_{\{x_0\}/\langle x_0 \rangle}$ for $\mathfrak{A}$ being constituted by the single quite transparent equation $((x_0 \vee x_1) \wedge (x_2 \vee x_3)) \lessapprox ((x_0 \wedge x_1) \vee (x_2 \wedge x_3))$, in which case the $\mho^{\varepsilon^+}_{\{x_0\}}$-disjunctivity of $\mathfrak{A}$ is easily seen immediately,/ by the following two (like in [7]) quite transparent equations:

$$
\begin{aligned}
(x_2 \vee x_3) &\lessapprox ((x_0 \vee x_1) \vee (x_2 \wedge x_3)),\\
((x_0 \wedge x_1) \wedge (x_2 \vee x_3)) &\lessapprox (x_2 \wedge x_3).
\end{aligned}
$$

On the other hand, recall that the majority term μ^+ for the two-element lattice $\mathfrak{D}_2$ is a dual discriminator for it, in which case the finite restricted disjunctivity/implicativity of $\mathbf{QV}(\mathfrak{A})$ equally (and constructively) ensues from Remark 3.38. However, the disjunctive/implicative system $\mho^{\vee/\supset}_{\mu^+}$, though being one-/two-element as well, is far more cumbersome and less transparent than $\mho^{\varepsilon^+}_{\{x_0\}/\langle x_0 \rangle}$ that highlights the value of Theorem 4.15 even in the most elementary case involved here.

In addition, since ε^+ is an equational implication for $\langle \mathfrak{A}, \{1\} \rangle$, the constructive proofs of Theorem 12(iii)⇒(i) of [24] and Lemma A.2 of [26] do yield (constructively as well) the restricted implicativity of $\mathbf{QV}(\mathfrak{A})$. However, the restricted congruence scheme arising in this way has 32 equations. The reduction factor 16, though being not especially impressing, definitely illustrates one of crucial advances of the present study with regard to [24] and [26].

And what is more, the next subsection provides an application of Theorem 4.15 void of the competing alternatives mentioned above *at all*.

6.2. **Four-valued expansions of De Morgan lattices.** Here, fix any signature $\Sigma \supseteq \Sigma_0 \triangleq (\Sigma^+ \cup \{\sim\})$, where $\sim$ is unary.

A *De Morgan lattice* (cf. [1], [8], [15], [20]) is any Σ_0-algebra $\mathfrak{A}$ such that $\mathfrak{A}{\restriction}\{\wedge, \vee\}$ is a distributive lattice and the following Σ_0-identities are true in $\mathfrak{A}$:

$$\sim\sim x_0 \approx x_0, \tag{6.1}$$

$$\sim(x_0 \vee x_1) \approx \sim x_0 \wedge \sim x_1, \tag{6.2}$$

$$\sim(x_0 \wedge x_1) \approx \sim x_0 \vee \sim x_1, \tag{6.3}$$

the variety of all them being denoted by DML.

By $\mathfrak{DM}_4$ we denote the De Morgan lattice such that $(\mathfrak{DM}_4{\restriction}\Sigma^+) \triangleq \mathfrak{D}_2^2$ and $\sim^{\mathfrak{DM}_4}\vec{a} \triangleq \langle 1 - a_{1-i} \rangle_{i \in 2}$, for all $\vec{a} \in 2^2$. In this connection, we use the following standard abbreviations:

$$
\begin{aligned}
\mathsf{t} &\triangleq \langle 1, 1 \rangle,\\
\mathsf{f} &\triangleq \langle 0, 0 \rangle,\\
\mathsf{b} &\triangleq \langle 1, 0 \rangle,\\
\mathsf{n} &\triangleq \langle 0, 1 \rangle.
\end{aligned}
$$

Consider any expansion $\mathfrak{A}$ of $\mathfrak{DM}_4$, that is, a Σ-algebra such that $(\mathfrak{A}\restriction\Sigma_0) = \mathfrak{DM}_4$. We start from recalling the following well-known fact with providing a canonical insight to it demonstrating applicability of both Subsection 6.1 and Corollary 3.30:

Lemma 6.3. *Every non-one element subalgebra $\mathfrak{B}$ of $\mathfrak{DM}_4$ is simple.*

Proof. Consider any $\theta \in (\mathrm{Con}(\mathfrak{B}) \setminus \{\Delta_B\})$, in which case $\theta \in \mathrm{Con}(\mathfrak{D}_2^2\restriction B)$. Let us show, by contradiction, that $\theta = B^2$. For suppose $\theta \neq B^2$. Then, by Subsection 6.1 and Corollary 3.30, $\theta = \ker(\pi_i\restriction B)$, for some $i \in 2$. Take any $\langle\vec{a},\vec{b}\rangle \in (\theta \setminus \Delta_B) \neq \varnothing$, in which case $\langle\sim^{\mathfrak{B}}\vec{a},\sim^{\mathfrak{B}}\vec{b}\rangle \in \theta$, and so both $a_i = b_i$ and $a_{1-i} = (1 - \pi_i(\sim^{\mathfrak{B}}\vec{a})) = (1 - \pi_i(\sim^{\mathfrak{B}}\vec{b})) = b_{1-i}$. This contradicts to the fact that $\vec{a} \neq \vec{b}$, as required. □

By Lemma 6.3 and Corollary 5.12, we then *immediately* conclude that $\mathsf{Q}(\mathfrak{A})$ is a restricted implicative (and so both semi-simple and finitely restricted disjunctive; cf. Theorem 3.25 and Remark 3.23(v), respectively) variety, its simple/[finitely-]subdirectly-irreducible members being exactly isomorphic copies of non-one-element subalgebras of $\mathfrak{A}$. This, in particular, explains why the quasivarieties generated by arbitrary expansions of $\mathfrak{DM}_4$ (including both itself and those miscellaneous ones studied in [19]) prove to be varieties. Another insight into this interesting fact demonstrating the applicability of Corollary 3.32 results from the latter and the following well-known result:

Proposition 6.4 (cf. [8], [19] and Proposition 3.2 of [20])**.** $\mathsf{DML} = \mathbf{PV}(\mathfrak{DM}_4)$.

Proof. With using Remark 2.4. For consider any $\mathfrak{B} \in \mathsf{DML}$ and any distinct $a, b \in B$. Then, $c \triangleq (a \vee^{\mathfrak{B}} b) \nleq^{\mathfrak{B}} d \triangleq (a \wedge^{\mathfrak{B}} b)$, so, by the Prime Ideal Theorem, there is some prime filter F_0 of $\mathfrak{B}\restriction\Sigma^+$ such that $d \notin F_0 \ni c$, in which case $(a \in F_0) \Leftrightarrow (b \notin F_0)$. Therefore, by (6.1), (6.2) and (6.3), $F_1 \triangleq (\sim^{\mathfrak{B}})^{-1}[B \setminus F_0]$ is a prime filter of $\mathfrak{B}\restriction\Sigma^+$ such that $\sim^{\mathfrak{B}}c \notin F_1 \ni \sim^{\mathfrak{B}}d$. Hence, by Claim 6.2, $h : B \to 2^2, e \mapsto \langle\chi_B^{F_i}(e)\rangle_{i\in 2}$ is a homomorphism from $\mathfrak{B}\restriction\Sigma^+$ to $\mathfrak{D}_2^2$. Then, by (6.1), it is routine checking that $h(\sim^{\mathfrak{B}}e) = \sim^{\mathfrak{DM}_4}h(e)$, for all $e \in B$. Thus, $h \in \hom(\mathfrak{B}, \mathfrak{DM}_4)$. Moreover, $(\pi_0(h(a)) = 1) \Leftrightarrow (\pi_0(h(a)) \neq 1)$, in which case $\pi_0(h(a)) \neq (\pi_0(h(a))$, and so $h(a) \neq h(b)$, as required. □

In particular, we get the well-known characterization of subdirectly-irreducible De Morgan lattices going back to [8] and also discussed in [27]. And what is more, Proposition 6.4 collectively with Lemma 3.36 eventually provide a new, generic and quite transparent insight into the issue of REDPC for DML going back to [27].

However, such argumentation of the finite restricted disjunctivity/implicativity of DML is not constructive. Nevertheless, $\{a, \mathsf{t}\}$, where $a \in \{\mathsf{b}, \mathsf{n}\}$, are exactly all prime filters of $\mathfrak{A}$, because any prime filter of it contains t but does not contain f, and so contains b iff it does not contain n, while $\widetilde{\Delta} \triangleq \{x_0, \sim x_0\}$ is an equality determinant for $\langle\mathfrak{A}, \{a, \mathsf{t}\}\rangle$, and so for $\mathfrak{A}$. Hence, by Theorem 4.15, we see that $\mathbf{QV}(\mathfrak{A})$ (in particular, DML; cf. Proposition 6.4) is restricted finitely $\mho^{\varepsilon^+}_{\widetilde{\Delta}/\langle x_0,\sim x_0\rangle}$-disjunctive/-implicative, the disjunctive/implicative system $\mho^{\varepsilon^+}_{\widetilde{\Delta}/\langle x_0,\sim x_0\rangle}$ for $\mathfrak{A}$ being constituted by the four quite transparent equations

$$((\sim^i x_0 \vee \sim^i x_1) \wedge (\sim^j x_2 \vee \sim^j x_3)) \lessapprox ((\sim^i x_0 \wedge \sim^i x_1) \vee (\sim^j x_2 \wedge \sim^j x_3)), \tag{6.4}$$

where $i, j \in 2$/(like in [27]):

$$\begin{aligned}
(x_2 \vee x_3) &\lessapprox ((((x_0 \vee \sim x_0) \vee x_1) \vee \sim x_1) \vee (x_2 \wedge x_3)),\\
((\sim x_0 \wedge \sim x_1) \wedge (x_2 \vee x_3) &\lessapprox ((x_0 \vee x_1) \vee (x_2 \wedge x_3)),\\
((x_0 \wedge x_1) \wedge (x_2 \vee x_3)) &\lessapprox ((\sim x_0 \vee \sim x_1) \vee (x_2 \wedge x_3)),\\
((((x_0 \wedge \sim x_0) \wedge x_1) \wedge \sim x_1) \wedge (x_2 \vee x_3)) &\lessapprox (x_2 \wedge x_3).
\end{aligned}$$

In this connection, it is remarkable that the constructive proofs of Theorem 12(iii)⇒ (i) of [24] and Lemma A.2 of [26] are not applicable to $\mathfrak{A}$, in case it is a reduction (in particular, is $\mathfrak{DM}_4$ itself) of the *Boolean De Morgan algebra* (cf. [21]) $\mathfrak{BDM}_4$, resulted from $\mathfrak{DM}_4$ by supplementing it with lattice bounds and the complement operation, because, in that case, the mirror permutation $\{\langle\langle i,j\rangle,\langle j,i\rangle\rangle \mid i,j \in 2\}$ of 2^2 is a non-singular non-diagonal endomorphism of $\mathfrak{A}$, and so, by Proposition 4.1, there is no matrix with underlying algebra $\mathfrak{A}$, equality determinant and equational implication. On the other hand, $\widetilde{\Delta}/\{(x_0 \wedge \sim x_1) \lessapprox x_1\}$ is an equality determinant/ equational implication for $\langle\mathfrak{DM}_4\restriction\{\mathsf{f},\mathsf{b},\mathsf{t}\}, \{\mathsf{b},\mathsf{t}\}\rangle$, while $\{\mathsf{b},\mathsf{t}\}$ is a prime filter of $\mathfrak{DM}_4\restriction\{\mathsf{f},\mathsf{b},\mathsf{t}\}$, in which case the constructive proofs of Lemma 11 and Theorem 12(iii)⇒(i) of [24] and Lemma A.2 of [26] are applicable to $\mathfrak{DM}_4\restriction\{\mathsf{f},\mathsf{b},\mathsf{t}\}$ and yield a restricted implicative system for it consisting of 2^{14} equations, and so the reduction factor $2^{12} = 4096$ more than well justifies the advance of the present study with regard to [24] and [26].

And what is more, Remark 3.38 is not applicable to $\mathfrak{DM}_4$ as well, as it follows from the negative results obtained below.

Let $\sqsubseteq$ be the partial ordering on 2^2 given by $(\vec{a} \sqsubseteq \vec{b}) \overset{\text{def}}{\Longleftrightarrow} ((a_0 \subseteq b_0)\&(b_1 \subseteq a_1))$, for all $\vec{a}, \vec{b} \in 2^2$. Then, an $f : B^n \to B$, where $n \in \omega$ and $B \subseteq 2^2$, is said to be *regular*, provided, for all $\bar{a}, \bar{b} \in B^n$ such that $a_i \sqsubseteq b_i$, for each $i \in n$, it holds that $f(\bar{a}) \sqsubseteq f(\bar{b})$. Clearly, any nullary operation on B is regular, for $\sqsubseteq$ is reflexive. Moreover, each operation of $\mathfrak{DM}_4$ is well known to be regular (cf., e.g., [19]), and so is any bilattice operation, for it is definable via operations of $\mathfrak{DM}_4$ and constants (cf. [19]). In this way, the following generic result equally covers $\mathfrak{DM}_4$ itself as well as both pure and bounded bilattice expansions of it:

Proposition 6.5. *Let $\Sigma \supseteq \Sigma_0$ and $\mathfrak{B}$ a Σ-algebra. Suppose $(\mathfrak{B}\restriction\Sigma_0) \in \mathbf{S}_{>1}\mathfrak{DM}_4$, $(\{\mathsf{n},\mathsf{b}\} \cap B) \neq \varnothing$ and each operation of $\mathfrak{B}$ is regular. Then, $\mathfrak{B}$ has no[dual] discriminator. In particular, any variety of Σ-algebras containing $\mathfrak{B}$ is not[dual] discriminator.*

Proof. By contradiction. For suppose some $\tau \in \mathrm{Tm}^3_\Sigma$ is a[dual] discriminator for $\mathfrak{B}$. Then, $\tau^{\mathfrak{B}} : B^3 \to B$ is regular. And what is more, since $(\mathsf{n} \vee^{\mathfrak{DM}_4} \mathsf{b}) = \mathsf{t}$, while $\sim^{\mathfrak{DM}_4}\langle i, i\rangle = \langle 1-i, 1-i\rangle$, for each $i \in 2$, whereas $|B| > 1$, $\{\mathsf{f}, \mathsf{t}\} \subseteq B$. Hence, as $B \ni (\mathsf{n}/\mathsf{b}) \sqsubseteq / \sqsupseteq \mathsf{f}$, we get $\mathsf{f}[\mathsf{t}] = \tau^{\mathfrak{B}}(\mathsf{f}, \mathsf{n}/\mathsf{b}, \mathsf{t}) \sqsubseteq / \sqsupseteq \tau^{\mathfrak{B}}(\mathsf{f}, \mathsf{f}, \mathsf{t}) = \mathsf{t}[\mathsf{f}]$. As expansions retain the subdirect irreducibility, this contradiction completes the argument. □

In view of Propositions 5.2 and 5.4 of [26], the condition of the regularity of operations of $\mathfrak{B}$ cannot be omitted in the formulation of Proposition 6.5. Likewise, $\mathfrak{DM}_4{\restriction}\{\mathsf{f}, \mathsf{t}\}$, being a two-element Boolean lattice, is well-known to have a [dual]discriminator (for instance, $\tau \triangleq ((x_{2[-2]} \wedge \varepsilon) \vee (x_{0[+2]} \wedge {\sim}\varepsilon))$, where $\varepsilon \triangleq (({\sim}x_0 \vee x_1) \wedge ({\sim}x_1 \vee x_0)))$. Therefore, the condition $(\{\mathsf{n}, \mathsf{b}\} \cap B) \neq \varnothing$ cannot be omitted in the formulation of Proposition 6.5 as well.

Let $\Sigma_{01} \triangleq \{\bot, \top\}$, where both $\bot$ and $\top$ are nullary. A *De Morgan algebra* (cf. [1]) is any $(\Sigma_0 \cup \Sigma_{01})$-algebra $\mathfrak{B}$ such that $\mathfrak{B}{\restriction}\Sigma_0$ is a De Morgan lattice and $\mathfrak{B}{\restriction}(\Sigma^+ \cup \Sigma_{01})$ is a bounded lattice with zero $\bot^{\mathfrak{B}}$ and unit $\top^{\mathfrak{B}}$, the variety of all them being denoted by DMA. A *Kleene lattice/algebra* is any De Morgan lattice/algebra satisfying the identity $(x_0 \wedge {\sim}x_0) \lessapprox (x_1 \vee {\sim}x_1)$, the variety of all them being denoted by KL/KA, respectively. Then, for each $a \in \{\mathsf{n}, \mathsf{b}\}$, $(\mathfrak{DM}_4{\restriction}\{\mathsf{f}, a, \mathsf{t}\}) \in \mathsf{KL}$. Therefore, by Proposition 6.5, we eventually get (the non-dual case of De Morgan/Kleene lattices has been due to Corollary 4.12 and Proposition 5.11 of [26]):

Corollary 6.6. $\mathsf{DML}/\mathsf{DMA}/\mathsf{KL}/\mathsf{KA}$ *is not[dual] discriminator.*

Thus, meanwhile, Theorem 4.15 remains a unique *generic constructive* tool of proving finite restricted disjunctivity of and REDPC for DML that highlights its power. And what is more, DML is an example of a variety generated by finitely many finite distributive lattice expansions with finite equality determinant (in particular, locally-finite implicative) non-dual-discriminator variety (among other things, it is such examples that highlight the non-triviality of the main result of [4] taking the formula "discriminator"=("dual discriminator"+"congruence-permutable"), being actually due to [12], into account). On the other hand, this subsection highlights the triviality of the elaboration of [27].

6.3. **Finitely-valued Łukasiewicz' algebras.** Let $\Sigma \triangleq (\Sigma_0 \cup \{\supset\})$, where $\supset$ is binary. Given any $n \in (\omega \setminus 2)$, by $\mathfrak{L}_n$ we denote the Σ-algebra such that $(\mathfrak{L}_n{\restriction}\Sigma^+) \triangleq \mathfrak{D}_n$, ${\sim}^{\mathfrak{L}_n} i \triangleq (n-1-i)$ and $(i \supset^{\mathfrak{L}_n} j) \triangleq \min(n-1-i+j, n-1)$, for all $i, j \in n$ (cf. [11] for the case $n = 3$). By induction on any $m \in (\omega \setminus 1)$, define the secondary unary $(m \otimes x_0) \in \mathrm{Tm}^1_\Sigma$ as follows (cf. Example 7 of [24]):

$$(m \otimes x_0) \triangleq \begin{cases} x_0 & \text{if } m = 1, \\ {\sim}x_0 \supset ((m-1) \otimes x_0) & \text{otherwise,} \end{cases}$$

in which case $(m \otimes^{\mathfrak{L}_n} i) = \min(n-1, m \cdot i)$, for all $i \in n$. Put $\gamma_n(x_0) \triangleq ((n-1) \otimes {\sim}x_0) \in \mathrm{Tm}^1_\Sigma$.

Proposition 6.7. *Let* $n \in (\omega \setminus 2)$. *Then,* $\mathfrak{L}_n$ *has an equality determinant with at most* $n-1$ *elements.*

Proof. By Example 3 of [23], there is an equality determinant Δ_n for $\langle \mathfrak{L}_n, \{n-1\}\rangle$ with at most $n-1$ elements (cf. Proposition 6.10 of [25] for a constructive proof of it). Then, $\Upsilon_n \triangleq \{\gamma_n(\delta) \mid \delta \in \Delta_n\} \subseteq \mathrm{Tm}^1_\Sigma$ has at most $n-1$ elements.

Consider any prime filter F of $\mathfrak{L}_n$, in which case $0 \notin F \ni (n-1)$, and so $(\gamma_n^{\mathfrak{L}_n})^{-1}[n \setminus F] = \{n-1\}$. In this way, Υ_n is an equality determinant for $\langle \mathfrak{L}_n, F\rangle$, as required. □

On the other hand, $\mathfrak{L}_n$ is well-known to have a discriminator (for instance, $\tau_n \triangleq ((x_2 \wedge \varepsilon_n) \vee (x_0 \wedge {\sim}\varepsilon_n))$, where $\varepsilon_n \triangleq {\sim}\gamma_n((x_0 \supset x_1) \wedge (x_1 \supset x_0)))$. For this reason, application of Theorem 4.15 to it looks not especially illustrative, in view of Proposition 6.7, especially because of polynomial growth of the upper bound the number $(n-1)^2/n \cdot (n-1)$ of equations in $\mho^{\varepsilon^+}_{\Upsilon_n}/\mho^{\varepsilon^+}_{\bar{\varphi},\mathfrak{L}_n}$, where $\bar{\varphi} : |\Upsilon_n| \to \Upsilon_n$ is any bijection, with increasing the number n of truth values, while the upper bound of the cardinality 2^{n-1} of $\mho^{\varepsilon^+}_{\bar{\varphi}}$ grows even exponentially, whereas Remark 3.38 yields a disjunctive/implicative system for it consisting of a single equation. However, this is nothing in comparison with the combinatorial growth of the upper bound of the cardinality $(2 \cdot (n-1))^{(2 \cdot n)+2}$ of the implicative system provided by the constructive proofs of Lemma 11 and Theorem 12(iii)⇒(i) of [24] and Lemma A.2 of [26]. Anyway, it is the previous subsection that has provided a really illustrative and *unique generic constructive* application of Theorem 4.15.

6.4. **Stone algebras.** A *Stone algebra* is any Σ_0-algebra $\mathfrak{A}$ such that $\mathfrak{A}{\restriction}\Sigma^+$ is a distributive lattice and $\mathfrak{A}$ satisfies the identities (6.2), (6.3) and the following ones:

$$(x_0 \wedge {\sim}x_0) \lessapprox x_1, \tag{6.5}$$

$$x_0 \lessapprox {\sim}{\sim}x_0, \tag{6.6}$$

the variety of all them being denoted by SA. (This axiomatization of Stone algebras is equivalent to the standard one resulted from adding (6.3) to the axiomatization of pseudo-complemented distributive lattices presented in [1], because the latter is easily seen to be derivable from the former, while the converse is well known to be true too.) A *Boolean algebra* is any Stone one satisfying the identity:

$$x_1 \lessapprox (x_0 \vee {\sim}x_0), \tag{6.7}$$

the variety of all them being denoted by BA.

By $\mathfrak{S}_3$ we denote the Stone algebra such that $(\mathfrak{S}_3{\restriction}\Sigma^+) \triangleq \mathfrak{D}_3$ and $({\sim}^{\mathfrak{S}_3} i) \triangleq \max\{j \in 3 \mid \min(i, j) = 0\}$. Likewise, by $\mathfrak{B}_2$ we denote the usual Boolean algebra with carrier 2. Then, $e_3 : 2 \to 3, i \mapsto (2 \cdot i)$ is an embedding of $\mathfrak{B}_2$ into $\mathfrak{S}_3$.

Proposition 6.8. $\mathsf{SA} = \mathbf{PV}(\mathfrak{S}_3)$.

Proof. With using Remark 2.4. For consider any $\mathfrak{A} \in \mathsf{SA}$ and any distinct $a, b \in F_0 \triangleq A$, in which case $c \triangleq (a \vee^{\mathfrak{A}} b) \nleqslant^{\mathfrak{A}} d \triangleq (a \wedge^{\mathfrak{A}} b)$, and so, by the Prime Ideal Theorem, there is some prime filter $d \notin F_2 \ni c$ of $\mathfrak{A}$, in which case $(a \in F_2) \Leftrightarrow (b \notin F_2)$. Then, by (6.2), (6.3), (6.5) and (6.6), $\sim^{\mathfrak{A}} c \notin F_1 \triangleq (\sim^{\mathfrak{A}})^{-1}[A \setminus F_2] \supseteq F_2$ is a prime filter of $\mathfrak{A}$, in which case, by Claim 6.2, $h \triangleq \chi^{\vec{F}} \in \hom(\mathfrak{A}{\upharpoonright}\Sigma^+, \mathfrak{D}_3)$. Moreover, by (6.5), it is routine checking that $h(\sim^{\mathfrak{A}} e) = \sim^{\mathfrak{S}_3} h(e)$, for all $e \in A$. Thus, $h \in \hom(\mathfrak{A}, \mathfrak{S}_3)$. Finally, $(h(a) = 2) \Leftrightarrow (h(b) \neq 2)$, and so $h(a) \neq h(b)$, as required. □

Next, $\mathbf{S}\mathfrak{S}_3 = \{\mathfrak{S}_3, \mathfrak{S}_3{\upharpoonright}(\operatorname{img} e_3)\}$, in which case, by Proposition 6.8 and (2.2), $\mathrm{SI}(\mathsf{SA}) \subseteq \mathbf{I}\{\mathfrak{S}_3, \mathfrak{B}_2\}$, and so, in view of Theorem 2.13, $\mathfrak{S}_3$, being non-Boolean, is subdirectly-irreducible. Moreover, $\mathfrak{S}_3{\upharpoonright}(\operatorname{img} e_3)$, being two-element, is simple, and so subdirectly irreducible. Therefore, by Corollary 3.21 with $M = \infty$ and Proposition 6.8, we conclude that SA is finitely restricted disjunctive, while its (finitely-)subdirectly-irreducibles are exactly isomorphic copies of either $\mathfrak{S}_3$ or $\mathfrak{B}_2$. This characterization of subdirectly-irreducible Stone algebras, though being well-known, is normally argued *ad hoc* in a much less transparent way (cf., e.g., [6]) that highlights the power of generic results obtained in Subsubsection 3.2.1. However, the above argumentation of the finite restricted disjunctivity of SA is not constructive. On the other hand, $\{2\}$ is a prime filter of $\mathfrak{S}_3$, while $\widetilde{\Delta}$ is an equality determinant for $\langle \mathfrak{S}_3, \{2\} \rangle$, in which case, by Remarks 4.2, 4.3, Theorem 4.8 and Proposition 6.8, we conclude that SA is finitely restricted $\mho^{\varepsilon^+}_{\widetilde{\Delta}}$-disjunctive, the disjunctive system $\mho^{\varepsilon^+}_{\widetilde{\Delta}}$ for $\mathfrak{S}_3$ being given by (6.4). Nevertheless, $\{\Delta_3, 3^2\} \not\ni (\Delta_3 \cup (3 \setminus 1)^2) \in \mathrm{Con}(\mathfrak{S}_3)$, in which case $\mathfrak{S}_3$ is not simple, and so has no equality determinant, in view of Lemma 4.14, while SA is not semi-simple. In particular, by Theorem 3.25, SA is not implicative. (In this way, SA becomes a representative instance of a finitely restricted disjunctive non-implicative variety with $\mathrm{Si}(\mathsf{SA})[\cup \mathbf{V}(\varnothing)]$ being the[universal] first-order model subclass of SA relatively axiomatized by the single[universal] first-order sentence $\Phi_{\leqslant 2} \wedge \Phi_{>1}$ [resp., $\Phi_{\leqslant 2}$], in view of Corollary 2.10, which shows that the condition of (relative)semi-simplicity cannot be omitted in the formulations of Theorems 5.2 and 5.17; a one more instance of such a kind is provided by the next subsection.) And what is more, since $\mathfrak{S}_3/\mathfrak{B}_2$ is embeddable into any non-Boolean/-one-element pseudo-complemented distributive lattice/Boolean algebra, while BA is discriminator, and so restricted implicative, in view of Remark 3.38, BA is thus the only non-trivial implicative quasivariety of pseudo-complemented distributive lattices, in view of Theorem 3.25. Among other things, $\mathfrak{S}_3$, being non-simple, has no[dual] discriminator, in which case SA is not[dual] discriminator. Therefore, Remark 3.38 is applicable to neither $\mathfrak{S}_3$ nor SA. Thus, meanwhile, Theorem 4.8 remains a unique and generic *constructive* tool of proving their finite restricted disjunctivity that highlights its power.

6.5. **HZ-algebras.** By $\mathfrak{H}\mathfrak{Z}$ we denote the Σ_0-algebra such that $(\mathfrak{H}\mathfrak{Z}{\upharpoonright}\Sigma^+) \triangleq \mathfrak{D}_3$ and $(\sim^{\mathfrak{H}\mathfrak{Z}} i) \triangleq (\min(i, 1) \cdot (3 - i))$, for all $i \in 3$. Members of $\mathsf{HZ} \triangleq \mathbf{QV}(\mathfrak{H}\mathfrak{Z}) = \mathbf{PV}(\mathfrak{H}\mathfrak{Z})$ [resp., $\mathsf{QHZ} \triangleq \mathbf{V}(\mathfrak{H}\mathfrak{Z})$] are referred to as *[quasi-]HZ-algebras* (cf. [22]). Note that $e'_3 : 2 \to 3, i \mapsto (i + 1)$ is an embedding of $\mathfrak{B}_2$ into HZ, so $\mathsf{BA} \subseteq \mathsf{HZ}$. And what is more, $\mathbf{S}_{>1}\mathfrak{H}\mathfrak{Z} = \{\mathfrak{H}\mathfrak{Z}, \mathfrak{H}\mathfrak{Z}{\upharpoonright}(\operatorname{img} e'_3)\}$.

Next, $\{2\}$ is a prime filter of $\mathfrak{H}\mathfrak{Z}$, while $\widetilde{\Delta}$ is an equality determinant for $\mathcal{HZ}' \triangleq \langle \mathfrak{H}\mathfrak{Z}, \{2\} \rangle$. Hence, by Remarks 4.2, 4.3 and Theorem 4.8, we conclude that HZ is finitely restricted $\mho^{\varepsilon^+}_{\widetilde{\Delta}}$-disjunctive, its HZ-(finitely-)subdirectly-irreducibles being exactly isomorphic copies of either $\mathfrak{H}\mathfrak{Z}$ or $\mathfrak{B}_2$, the disjunctive system $\mho^{\varepsilon^+}_{\widetilde{\Delta}}$ for $\mathfrak{H}\mathfrak{Z}$ being given by (6.4).

Let $\mathfrak{H}$ be the Σ_0-algebra such that $(\mathfrak{H}{\upharpoonright}\Sigma^+) \triangleq \mathfrak{D}_2$ and $\sim^{\mathfrak{H}} \triangleq \Delta_2$. Then, $h_2 : 3 \to 2, i \mapsto \min(i, 1)$ is a surjective homomorphism from $\mathfrak{H}\mathfrak{Z}$ onto $\mathfrak{H}$.

Lemma 6.9. $\mathrm{Con}(\mathfrak{H}\mathfrak{Z}) = \{\Delta_3, 3^2, \ker h_2\}$. *In particular,* $\mathfrak{H}\mathfrak{Z}$ *is subsirectly irreducible but is not simple.*

Proof. Consider any $\theta \in (\mathrm{Con}(\mathfrak{H}\mathfrak{Z}) \setminus \{\Delta_3, 3^2\})$, in which case $\theta \in \mathrm{Con}(\mathfrak{D}_3)$. Clearly, $e''_3 : 3 \to 2^2, i \mapsto \langle i - [i/2], [i/2] \rangle$ is an embedding of $\mathfrak{D}_3$ into $\mathfrak{D}_2^2$. Therefore, by Subsection 6.1, Corollary 3.30 and Lemma 2.8, we see that $\theta = \theta_j \triangleq \ker(\pi_j \circ e''_3)$, for some $j \in 2$. On the other hand, $\langle 0, 1 \rangle \in \theta_1$, while $\langle \sim^{\mathfrak{H}\mathfrak{Z}} 0, \sim^{\mathfrak{H}\mathfrak{Z}} 1 \rangle = \langle 0, 2 \rangle \notin \theta_1$, in which case $\theta_1 \notin \mathrm{Con}(\mathfrak{H}\mathfrak{Z})$, and so θ may be equal to θ_0 alone. And what is more, $\theta_0 = (\ker h) \in \mathrm{Con}(\mathfrak{H}\mathfrak{Z})$, while $\Delta_3 \neq \theta_0 \neq 3^2$, for $\langle 0, 1 \rangle \notin \theta_0 \ni \langle 2, 1 \rangle$. This completes the argument. □

Since any two-element algebra is simple, and so, in particular, subdirectly irreducible, by the Homomorphism Theorem, the congruence-distributivity of lattice expansions (cf. [17]), Theorems 2.13, 3.25, Corollaries 2.10, 2.15 and Lemma 6.9, we first get:

Corollary 6.10. $\mathrm{SI}^{[\omega]}(\mathsf{QHZ}) = \mathbf{I}\{\mathfrak{H}\mathfrak{Z}, \mathfrak{B}_2, \mathfrak{H}\}$ *and* $\mathrm{Si}(\mathsf{QHZ}) = \mathbf{I}\{\mathfrak{B}_2, \mathfrak{H}\}$. *In particular,* $\mathsf{QHZ} = \mathbf{Q}(\{\mathfrak{H}\mathfrak{Z}, \mathfrak{H}\})$ *is not semi-simple, and so is not implicative.*

On the other hand, $\{1\}$ is a prime filter of $\mathfrak{H}$, while $\{x_0\}$ is an equality determinant for $\langle \mathfrak{H}, \{1\} \rangle$, and so is $\widetilde{\Delta}$. Hence, by Corollary 6.10, Remarks 4.2, 4.3 and Theorem 4.8, we conclude that QHZ is finitely restricted $\mho^{\varepsilon^+}_{\widetilde{\Delta}}$-disjunctive. In this way, as two-element algebras are simple, by Corollary 6.10, QHZ is a one more instance of a finitely restricted disjunctive non-implicative variety with $\mathrm{Si}(\mathsf{QHZ})[\cup \mathbf{V}(\varnothing)]$ being the[universal] first-order model subclass of QHZ relatively axiomatized by the single[universal] first-order sentence $\Phi_{\leqslant 2} \wedge \Phi_{>1}$ [resp., $\Phi_{\leqslant 2}$].

Proposition 6.11. $\mathfrak{H}\mathfrak{Z}$ *is* HZ*-simple. In particular,* HZ *is relatively semi-simple but is not a variety.*

Proof. By the Homomorphism Theorem, $\mathfrak{H}\mathfrak{Z}/(\ker h_2)$ is isomorphic to $\mathfrak{H}$. However, $\mathfrak{H} \notin \mathsf{HZ}$, because the quasi-identity $(x_0 \approx \sim x_0) \to (x_0 \lessapprox x_1)$, being true in $\mathfrak{H}\mathfrak{Z}$, is not so in $\mathfrak{H}$ under $[x_i/(1 - i)]_{i \in 2}$. In this way, Lemma 6.9, Corollary 2.10 and the fact that $\mathfrak{B}_2$, being two-element, is simple, and so HZ-simple, complete the argument. □

Thus, in view of Proposition 6.11, Corollaries 3.21/5.12 are not applicable to proving the restricted finite disjunctivity/implicativity of HZ in principle, simply because they deal with the solely equational framework. Nevertheless, by Theorem

5.13 and Proposition 6.11, we do eventually conclude that HZ is restricted implicative. However, this argumentation is not constructive. On the other hand, we have:

Proposition 6.12. $\varepsilon^{\mathrm{HZ}} \triangleq \{x_0 \lessapprox (\sim x_0 \vee x_1)\}$ *is an equational implication for $\mathcal{HZ}'$.*

Proof. Consider any $a, b \in 3$. In case $b = 2$, we clearly have $(\sim^{\mathfrak{H3}} a \vee^{\mathfrak{H3}} b) = 2 \geqslant^{\mathfrak{H3}} a$. Likewise, in case $a \neq 2$, we have $a \leqslant^{\mathfrak{H3}} \sim^{\mathfrak{H3}} a \leqslant^{\mathfrak{H3}} (\sim^{\mathfrak{H3}} a \vee^{\mathfrak{H3}} b)$. Finally, assume both $a \leqslant^{\mathfrak{H3}} (\sim^{\mathfrak{H3}} a \vee^{\mathfrak{H3}} b)$ and $a = 2$, in which case $(1 \vee^{\mathfrak{H3}} b) = 2$, and so $b = 2$, as required. □

Thus, by Proposition 6.12 and Theorem 4.10, HZ is $\mho^{\varepsilon^{\mathrm{HZ}},\widetilde{\Delta}}_{\langle x_0,\sim x_0\rangle}$-implicative, the implicative system $\mho^{\varepsilon^{\mathrm{HZ}},\widetilde{\Delta}}_{\langle x_0,\sim x_0\rangle}$ for $\mathfrak{H3}$ being constituted by the following eight quite transparent equations:

$$\begin{aligned}
(\sim^k x_2 \vee \sim^k x_3) &\lessapprox ((\sim(\sim^k x_2 \vee \sim^k x_3) \\
&\quad \vee ((((x_0 \vee \sim x_0) \vee x_1) \vee \sim x_1))) \\
&\quad \vee (\sim^k x_2 \wedge \sim^k x_3)), \\
((\sim x_0 \wedge \sim x_1) \wedge (\sim^k x_2 \vee \sim^k x_3)) &\lessapprox (\sim((\sim x_0 \wedge \sim x_1) \wedge (\sim^k x_2 \vee \sim^k x_3)) \\
&\quad \vee ((x_0 \vee x_1) \vee (\sim^k x_2 \wedge \sim^k x_3))), \\
((x_0 \wedge x_1) \wedge (\sim^k x_2 \vee \sim^k x_3)) &\lessapprox (\sim((x_0 \wedge x_1) \wedge (\sim^k x_2 \vee \sim^k x_3)) \\
&\quad \vee ((\sim x_0 \vee \sim x_1) \vee (\sim^k x_2 \wedge \sim^k x_3))), \\
((((x_0 \wedge \sim x_0) \wedge x_1) \wedge \sim x_1) \wedge (\sim^k x_2 \vee \sim^k x_3)) &\lessapprox (\sim((((x_0 \wedge \sim x_0) \wedge x_1) \wedge \sim x_1) \\
&\quad \wedge (\sim^k x_2 \vee \sim^k x_3)) \vee (\sim^k x_2 \wedge \sim^k x_3)),
\end{aligned}$$

where $k \in 2$. In this connection, it is remarkable that Subsection 5.5 of [26] dealt with the matrix $\mathcal{HZ} \triangleq \langle \mathfrak{H3}, \{0, 2\} \rangle$, for which $\widetilde{\Delta}$ is an equality determinant too, and an equational implication for it, being, among other things, more cumbersome and less transparent than $\varepsilon^{\mathrm{HZ}}$. And what is more, $\{0, 2\}$ is not a prime filter of $\mathfrak{H3}$, in which case Theorem 4.10 is not applicable to $\mathcal{HZ}$, and so the only way to prove *constructively* the restricted implicativity of HZ therein was involving the constructive proofs of Theorems 12(iii)⇒(i) of [24] and Lemma A.2 of [26] that did yield a restricted implicative system for $\mathfrak{H3}$ consisting of 2^{14} equations. The reduction factor $2^{11} = 2048$ more than well justifies the advance of the present study with regard to [24] and [26].

Finally, $\mathfrak{H3}$, being non-simple, has no [dual]discriminator, in which case Remark 3.38 is not applicable to it, and so, meanwhile, Theorems 4.8/4.10 remain unique *generic constructive and effective* tools of proving the restricted finite disjunctivity/implicativity of HZ that highlights their power.

In general, it is such examples that justify the *quasi*equational framework accepted here. A one more instance of such a kind is discussed in the last subsection.

6.6. **Semilattices.** The variety of all semilattices, viewed as $\Sigma^{+}_{\frac{1}{2}}$-algebras, is denoted by SL. By $\mathfrak{S}_2$ we denote the semilattice over the poset 2 ordered by inclusion.

Proposition 6.13. $\mathsf{SL} = \mathbf{PV}(\mathfrak{S}_2)$.

Proof. With using Remark 2.4. For consider any $\mathfrak{A} \in \mathsf{SL}$ and any $\vec{a} \in (A^2 \setminus \Delta_A)$, in which case $a_i \nleqslant^{\mathfrak{A}} a_{1-i}$, for some $i \in 2$. Then, $a_{1-i} \notin F \triangleq \{a \in A \mid a_i \leqslant^{\mathfrak{A}} a\} \ni a_i$ is a filter of $\mathfrak{A}$. Therefore, $h \triangleq \chi^F_A \in \hom(\mathfrak{A}, \mathfrak{S}_2)$. Finally, $h(a_i) = 1 \neq 0 = h(a_{1-i})$, as required. □

Then, by (2.2) and Proposition 6.13, $\mathrm{SI}(\mathsf{SL}) \subseteq \mathbf{IS}_{>1}\mathfrak{S}_2$. On the other hand, any two-element algebra is simple, and so subdirectly-irreducible, and has no proper non-one-element subalgebra. Thus, SL is semi-simple and $\mathrm{Si}(\mathsf{SL}) = \mathbf{I}\mathfrak{S}_2$ consists exactly of two-element semilattices, in which case $\mathrm{Si}(\mathsf{SL})[\cup\mathbf{V}(\varnothing)]$ is the[universal] first-order model subclass of SL relatively axiomatized by the single[universal] first-order sentence $\Phi_{\leqslant 2} \wedge \Phi_{>1}$ [resp., $\Phi_{\leqslant 2}$]. Nevertheless, we have the following negative result:

Proposition 6.14. $\mathfrak{S}_2^2$ *is not congruence-modular, and so neither congruence-permutable nor congruence-distributive. In particular,* SL *is neither congruence-permutable nor(directly) congruence-modular, and so not(directly) congruence-distributive, in which case it is neither restricted disjunctive nor implicative, and so does not have EDPC.*

Proof. Clearly, $\hbar$ is an endomomorphism of $\mathfrak{S}_2^2$. Finally, Theorems 3.10, 3.37, 5.17 and Corollary 2.9 complete the argument. □

Proposition 6.14 shows that the condition of [relative](direct)congruence-distributivity cannot be omitted in the formulations of Theorems 5.2 and 5.17. And what is more, SL becomes a representative instance of a semi-simple non-implicative variety with (universally)first-order-axiomatizable class of all its simple(and one-element) members.

6.7. **Sette algebras.** By $\mathfrak{P}_1$ we denote the Σ_0-algebra with carrier 3 and operations defined as follows: for all $a, b \in 3$, put $\sim^{\mathfrak{P}_1} a \triangleq (2 \cdot (1 - \max(0, a-1)))$ and $(a \wedge [\vee]^{\mathfrak{P}_1} b) \triangleq (2 \cdot \min(\min[\max](a,b), 1))$. We also deal with the secondary unary operation $(\Diamond x_0) \triangleq (x_0 \wedge x_0)$. According to [18], members of $\mathsf{P1} \triangleq \mathbf{QV}(\mathfrak{P}_1)$ [resp., $\mathsf{QP1} \triangleq \mathbf{V}(\mathfrak{P}_1)$] are referred to as *[quasi-]Sette-algebras.* Then e_3 is an embedding of $\mathfrak{B}_2$ into $\mathfrak{P}_1$, so $\mathsf{BA} \subseteq \mathsf{P1}$. Moreover,

$$((\operatorname{img} f^{\mathfrak{P}_1}) \subseteq \{0, 2\}, \tag{6.8}$$

for every $f \in \Sigma_0$. Therefore, as $(\operatorname{img} e_2) = \{0, 2\}$ and $\sim^{\mathfrak{P}_1}(2 \cdot i) = (2 \cdot (1-i))$, for each $i \in 2$, we have $\mathbf{S}\mathfrak{P}_1 = \{\mathfrak{P}_1, \mathfrak{P}_1 \restriction (\operatorname{img} e_2)\}$.

Let $\mathfrak{Q}$ be the Σ_0-algebra with carrier 2 such that $(\operatorname{img} f^{\mathfrak{Q}}) \subseteq \{0\}$, for all $f \in \Sigma_0$. Then, in view of (6.8), $h_2' : 3 \to 2, i \mapsto \min(i, 2-i)$ is a surjective homomorphism from $\mathfrak{P}_1$ onto $\mathfrak{Q}$.

Lemma 6.15. $\operatorname{Con}(\mathfrak{P}_1) = \{\Delta_3, 3^2, \ker h_2'\}$. *In particular,* $\mathfrak{P}_1$ *is subsirectly irreducible but is not simple, in which case* $\mathsf{QP1}$ *is not semi-simple, and so is not implicative.*

Proof. Consider any $\theta \in (\operatorname{Con}(\mathfrak{P}_1) \setminus \{\Delta_3, 3^2\})$. Take any $\langle a, b\rangle \in (\theta \setminus \Delta_3)$. Consider the following complementary cases:

(1) $1 \notin \{a, b\}$.
Then, $\{a, b\} = \{0, 2\}$, in which case $\langle 0, 2\rangle \in \theta$.

(2) $1 \in \{a, b\}$.
Then, $\{a, b\} = \{1, 2 \cdot i\}$, for some $i \in 2$, in which case $\langle 1, 2 \cdot i\rangle \in \theta$, and so both $\langle 2, 2 \cdot (1-i)\rangle = \langle \sim^{\mathfrak{P}_1} 1, \sim^{\mathfrak{P}_1}(2 \cdot i)\rangle \in \theta$ and $\langle 2, 2 \cdot i\rangle = \langle \Diamond^{\mathfrak{P}_1} 1, \Diamond^{\mathfrak{P}_1}(2 \cdot i)\rangle \in \theta$, in which case $\langle 2 \cdot i, 2 \cdot (1-i)\rangle \in \theta$, and so $\langle 0, 2\rangle \in \theta$.

Thus, anyway, $\langle 0, 2\rangle \in \theta$, in which case $\theta \supseteq (\{0,2\}^2 \cup \Delta_3) = (\ker h_2') \in \max(\operatorname{Con}(\mathfrak{P}_1) \setminus \{3^2\})$, in view of Lemma 2.8, for $\mathfrak{Q}$, being two-element, is simple, and so $\theta = (\ker h_2')$. In this way, Theorem 3.25 completes the argument. ☐

Due to [the proof of]Theorem 5.7 of [18], it has already been known that $\mathrm{SI}^{[\omega]}(\mathsf{QP1}) = \mathbf{I}\{\mathfrak{P}_1, \mathfrak{B}_2, \mathfrak{Q}\}$. Hence, as simple algebras are subdirectly irreducible, while two-element algebras are simple, by Corollary 2.10 and Lemma 6.15, we first get:

Corollary 6.16. $\mathrm{Si}(\mathsf{QP1}) = \mathbf{I}\{\mathfrak{B}_2, \mathfrak{Q}\}$. *In particular,* $\mathrm{Si}(\mathsf{QP1})[\cup \mathbf{V}(\varnothing)]$ *is the[universal] first-order model subclass of* $\mathsf{QP1}$ *relatively axiomatized by the single[universal] first-order sentence* $\Phi_{\leqslant 2} \wedge \Phi_{>1}$ *[resp.,* $\Phi_{\leqslant 2}$*].*

Then, as opposed to SA and QHZ, we have:

Lemma 6.17. $\mathfrak{Q}^2$ *is not congruence-modular.*

Proof. Since, by (6.8), we have $(\operatorname{img} f^{\mathfrak{A}}) \subseteq \{\langle 0, 0\rangle\}$, for every $f \in \Sigma_0$, we see that $\hbar$ is an endomomorphism of $\mathfrak{Q}^2$. Then, Corollary 2.9 completes the argument. ☐

Proposition 6.18. $\mathfrak{P}_1^2$ *is not congruence-modular, and so neither congruence-permutable nor congruence-distributive. In particular,* $[\mathsf{Q}]\mathsf{P1}$ *is neither congruence-permutable nor(directly) congruence-modular, and so not(directly) congruence-distributive, in which case* $\mathsf{QP1}$ *is not restricted disjunctive.*

Proof. Clearly, $g : 3^2 \to 2^2, \langle i, j\rangle \mapsto \langle h_2'(i), h_2'(j)\rangle$ is a surjective homomorphism from $\mathfrak{P}_1^2$ onto $\mathfrak{Q}^2$. In this way, Lemmas 2.8, 6.15, 6.17 and Theorem 3.10 complete the argument. ☐

Taking Lemma 6.15 and Proposition 6.18 into account, we see that $\mathsf{QP1}$ combines negative features of both SL and either SA or QHZ.

And what is more, in view of [17] (more specifically, the congruence-distributivity of lattice expansions), Proposition 6.18 makes the case under consideration essentially beyond the lattice framework, and so both Remark 4.3, Subsubsection 4.2.1 and Corollaries 3.21 and 5.12, both corollaries being equally declined by Proposition 6.20 below.

Nevertheless, $\{1, 2\}$ is a prime filter of $\mathfrak{P}_1$, while $\widetilde{\Delta}$ is an equality determinant for $\mathcal{P}_1 \triangleq \langle \mathfrak{P}_1, \{1, 2\}\rangle$. And what is more, we have:

Proposition 6.19. $\varepsilon^{\mathrm{P1}} \triangleq \{\Diamond x_0 \lessapprox x_1\}$ *is an equational implication for* $\mathcal{P}_1$.

Proof. Consider any $a, b \in 3$. First, we have $\Diamond^{\mathfrak{P}_1} 0 = 0 = (0 \wedge^{\mathfrak{P}_1} b)$. Next, in case $b \neq 0$, we also have $(a \wedge^{\mathfrak{P}_1} b) = \Diamond^{\mathfrak{P}_1} a = \Diamond^{\mathfrak{P}_1} \Diamond^{\mathfrak{P}_1} a$. Finally, assume both $a \neq 0$, in which case $\Diamond^{\mathfrak{P}_1} a = 1$, and $(1 \wedge^{\mathfrak{P}_1} b) = 1$, in which case $b \neq 0$, as required. ☐

Thus, by Remark 4.4, Theorem 4.8 and Proposition 6.19, we conclude that $\mathsf{P1}$ is finitely restricted $\mho^{\varepsilon^{\mathrm{P1}}}_{\widetilde{\Delta}}$-disjunctive, its $\mathsf{P1}$-(finitely-)subdirectly-irreducibles being exactly isomorphic copies of either $\mathfrak{P}_1$ or $\mathfrak{B}_2$, the disjunctive system $\mho^{\varepsilon^{\mathrm{P1}}}_{\widetilde{\Delta}}$ for $\mathfrak{P}_1$ being constituted by the four quite transparent equations $\Diamond((\sim^i x_0 \vee \sim^i x_1) \wedge (\sim^j x_2 \vee \sim^j x_3)) \lessapprox ((\sim^i x_0 \wedge \sim^i x_1) \vee (\sim^j x_2 \wedge \sim^j x_3))$, where $i, j \in 2$.

Proposition 6.20. $\mathfrak{P}_1$ *is* $\mathsf{P1}$*-simple. In particular,* $\mathsf{P1}$ *is relatively semi-simple but is not a variety.*

Proof. By the Homomorphism Theorem, $\mathfrak{P}_1/(\ker h_2')$ is isomorphic to $\mathfrak{Q}$. However, $\mathfrak{Q} \notin \mathsf{P1}$, because the quasi-identity $\{\sim x_0 \approx \sim x_1, \Diamond x_0 \approx \Diamond x_1\} \to (x_0 \approx x_1)$, being true in $\mathfrak{P}_1$, is not so in $\mathfrak{Q}$ under $[x_i/i]_{i \in 2}$. In this way, Lemma 6.15, Corollary 2.10 and the fact that $\mathfrak{B}_2$, being two-element, is simple, and so $\mathsf{P1}$-simple, complete the argument. ☐

Thus, in view of Proposition 6.20, Corollaries 3.21/5.12 are not applicable to proving the restricted finite disjunctivity/implicativity of $\mathsf{P1}$ in principle, simply because they deal with the solely equational framework. Nevertheless, by Theorem 5.13 and Proposition 6.20, we do eventually conclude that $\mathsf{P1}$ is restricted implicative. However, this argumentation is not constructive. On the other hand, by Remark 4.4, Proposition 6.19 and Theorem 4.10, P1 is $\mho^{\varepsilon^{\mathrm{P1}},\tilde{\Delta}}_{\langle x_0,\sim x_0\rangle}$-implicative, the implicative system $\mho^{\varepsilon^{\mathrm{P1}},\tilde{\Delta}}_{\langle x_0,\sim x_0\rangle}$ for $\mathfrak{P}_1$ being constituted by the following eight quite transparent equations:

$$\begin{aligned}
\Diamond(\sim^k x_2 \vee \sim^k x_3) &\lessapprox ((((x_0 \vee \sim x_0) \vee x_1) \vee \sim x_1) \\
&\qquad \vee (\sim^k x_2 \wedge \sim^k x_3)),\\
\Diamond((\sim x_0 \wedge \sim x_1) \wedge (\sim^k x_2 \vee \sim^k x_3)) &\lessapprox ((x_0 \vee x_1) \vee (\sim^k x_2 \wedge \sim^k x_3)),\\
\Diamond((x_0 \wedge x_1) \wedge (\sim^k x_2 \vee \sim^k x_3)) &\lessapprox ((\sim x_0 \vee \sim x_1) \vee (\sim^k x_2 \wedge \sim^k x_3)),\\
\Diamond((((x_0 \wedge \sim x_0) \wedge x_1) \wedge \sim x_1) \wedge (\sim^k x_2 \vee \sim^k x_3)) &\lessapprox (\sim^k x_2 \wedge \sim^k x_3),
\end{aligned}$$

where $k \in 2$, while that, which results from the constructive proofs of Theorem 12(iii)⇒(i) of [24] and Lemma A.2 of [26], has 2^{14} equations, so, like for HZ-algebras, we equally reach the reduction factor 2^{11}.

On the other hand, $\mathfrak{P}_1$, being non-simple, has no [dual]discriminator, in which case Remark 3.38 is not applicable to it, and so, meanwhile, Theorems 4.8/4.10 remain unique generic tools of constructive proving the restricted finite disjunctivity/implicativity of P1 that highlights their power.

7. Conclusions

First of all, we should like to highlight that the purpose of this study was neither repairing nor quasi-equational relativizing [3] but providing both a new *and transparent* insight into the equivalence of restricted implicativity and [subsirect]filtrality *void of extra links* (like ideality, REDPC, RCEP, dually Browerian semilattices and generalized Boolean algebras becoming practically useless in this connection, especially because, though congruence-distributivity *immediately* ensues from the subdirect filtrality, in view of Theorem 3.6, congruence-generalized-Booleanity could hardly do so) and a proper parameterization of it (being essentially beyond the scopes of [3] *at all*) as well as really relevant connections between restricted implicativity and restricted disjunctivity advancing those immediate ones which have already been traced in [26].

After all, this paper has definitely shown that [relative](sub)direct filtrality is a right intrinsic characteristic feature of (restricted)implicativity very much like [relative]ideality for (sub)direct products is that of (R)EDP[R]C (cf. [3]).

And what is more, this paper has definitely shown that the conception of equality determinant initially introduced in [23] just for the sake of construction of propositional *two-side* sequent calculi (like *LK*; cf. [5]) with Cut Elimination Property for finitely-many-valued logics is valuable within the context of not merely General Logic but equally Universal Algebra.

Perhaps, the main problem remaining still open within this study is whether parameterized implicative [quasi]varieties have EDP[R]C. In view of [3], within the equational framework, it is equivalent to the question whether direct filtrality implies ideality for direct products. It would be equally interesting to elucidate whether distributive lattice expansions with [dual]discriminator have a (finite)equality determinant (just remind that both distributive lattices and finitely-valued Łukasiewicz' algebras as well as discriminator expansions of De Morgan lattices found in [26] do so).

And what is more, here, we have restricted our consideration by merely restricted (viz., non-parameterized) disjunctivity. The thing is that the issue of parameterized one needs further advanced investigations to be eventually presented elsewhere.

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