A Game of Life on a Pythagorean Tessellation

Soorya Annadurai

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Soorya Annadurai
Microsoft R&D Pvt. Ltd.
Bangalore, India
soorya.annadurai@gmail.com

***Abstract*—Conway's Game of Life is a zero-player game played on an infinite square grid, where each cell can be either "dead" or "alive". The interesting aspect of this game arises when we observe how each cell interacts with its neighbors over time. There has been much public interest in this game, and several variants have become popular. Research has shown that similar Games of Life can exist on hexagonal, triangular, and other tiled grids. Games of Life have also been devised in 3 dimensions. In this work, another Game of Life is proposed that utilizes a Pythagorean tessellation, with a unique set of rules. Several interesting life forms in this universe are also illustrated.**

***Index Terms*—cellular automata, Game of Life, two-square tiling, Pythagorean tessellation**

I. Introduction

Carter Bays defines a cellular automaton [3] as a structure comprising a grid with individual cells that can have two or more states; these cells evolve in discrete time units and are governed by a rule, which usually involves neighbors of each cell. There are many real-world applications of cellular automata[8], including VLSI design, pseudo-random number generation, cryptography, image compression, and many others.

A tessellation or tiling is composed of a specific shape that is repeated endlessly in a plane, with no gaps or overlaps. Examples of simple tessellations are the square grid, the triangular grid (a plane completely covered by identical triangles), etc. In this work, the word "grid" is interchangeable with "tiling" or "tesselation".

Conway's Game of Life is a classic example of a cellular automata on a square grid. The rules of Conway's Game of Life are delightfully simple. They was first published[7] to the public in 1970. Because of its analogies with the rise, fall and alternations of a society of living organisms, it belongs to a growing class of what are called "simulation games" - games that resemble real-life processes. The game is played on an infinite square grid. The basic idea is to start with a configuration of states (*dead* or *alive*), one per cell, then observe how it changes as Conway's "genetic laws" for births, deaths, and survival are applied. These genetic laws are as follows:

1) Survivals. Every living cell with two or three neighboring living cell survives for the next generation.
2) Deaths. Each living cell with four or more neighbors dies (is removed) from overpopulation. Every living cell with one neighbor or none dies from isolation.
3) Births. Each empty cell adjacent to exactly three neighbors is a birth cell. It will become alive in the next generation.

In a cellular automaton of this type, a single cell may do one of four things within a single time step[6]: If it was dead but becomes alive, we say that it is born. If it was alive and remains alive, we say that it survives. If it was alive and becomes dead, we say that it dies. And if it was dead and remains dead, we say that it is quiescent. These are all the state transitions possible by a single cell.

It is important to understand that all transitions (like births and deaths) occur simultaneously. Together, they constitute a single generation.

This combination of genetic laws is referred to as a [Game of Life] rule.

II. Literature Survey

Excitement and popular interest into Conway's Game of Life started when Martin Gardner published this Scientific American article[7], first introducing the rules of this zero-player game to the public and responding to one of Conway's challenges. (These rules are explained in Section I. The challenge was to either prove or disprove Conway's conjecture that no pattern can expand without limit. This was disproved by the discovery of a "gun", which is elaborated in section II-B.)

A. The Speed of Light

The *speed of light* on regularly tiled square grid is defined as the speed of a chess king moving in any direction[7]. To make definition this more generic to other tiling patterns, it can be defined as the furthest distance in one generation that neighbors a tile. This is because it is the highest speed at which any kind of movement can occur on the board. It is also referred to as c.

B. Categories of Life

After this, there have been many public attempts to discover "life forms" that abide by Conway's rules. They can be broadly classified[10] into still life, oscillators, gliders, guns, and puffer trains. Still life patterns do not change from one generation to the next; oscillators return to their initial state after a finite number of generations; gliders translate themselves across the grid; guns indefinitely generate patterns that translate across the grid; puffer trains translate across the grid, leaving behind debris that don't eventually disappear on their own.

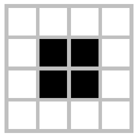
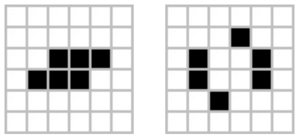


Fig. 1: A still-life form in Conway's Game of Life - the 'Block'

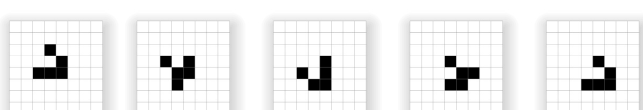


(a) (b)

Fig. 2: An oscillator's states in Conway's Game of Life - the 'Toad'

Figure 1 depicts a still-life form in Conway's Game of Life. In the next generation of this configuration, no living cells die, and no dead cells come alive. Figure 2a depicts an oscillator. The next generation (fig. 2b) changes, but the subsequent generation returns to its original state. Therefore, it has a periodicity of 2. (Extrapolating on this concept, still-life forms can also be considered as oscillators[10] with a periodicity of 1.) Figure 3 depicts a glider. 4 generations later (in fig. 3e), the same pattern from the original configuration is repeated, but it has translated by 1 tile to the bottom-right. Therefore, it's this glider's speed is $c/4$. Figure 4 depicts a gun that shoots out gliders with the configuration in fig. 3. It is called the "Gosper glider gun"; it produces its first glider on the 15th generation, and another glider every 30th generation from then on. This pattern was discovered by Bill Gosper's team from Massachusetts Institute of Technology, and is proof that a pattern in Conway's Game of Life can grow without limit.

There are many other patterns of still life, oscillators, gliders, and guns that have been discovered[4] in the original Conway's Game of Life.



(a) (b) (c) (d) (e)

Fig. 3: An glider's states in Conway's Game of Life

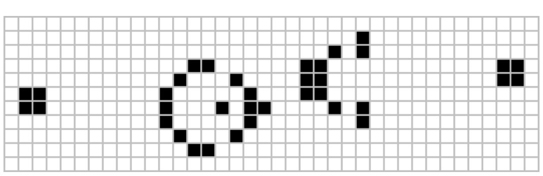


Fig. 4: A glider gun in Conway's Game of Life - the 'Gosper Gun'

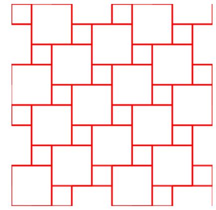


Fig. 5: A Pythagorean tessellation, also known as a two-square tiling

C. *Requirements for a valid Game of Life*

A valid Game of Life rule must satisfy 3 criteria[2].

1) When counting the neighbors of a cell, all touching neighbors are considered and treated the same.
2) At least one glider exists.
3) Start with a finite wrapped universe that is completely filled with a random pattern. Then after a finite number of generations, all such patterns eventually must either disappear, or decompose into one or more oscillators. Rules exhibiting this property are said to be stable.

D. *Variants of Conway's Game of Life*

There are many valid variants of the original Conway's Game of Life. Some variants involve using the same regularly tiled square grid, but having a different set of rules for survival, deaths, and births[4, 1]. For example, there is a valid variant where living cells survive if they have exactly 3 OR 4 living neighbors (and die if they don't), and dead cells are reborn if they have exactly 3 OR 4 living neighbors[1].

It is also possible to use different types of grid. Carter Bays describes similar cellular automata in triangular, pentagonal, and hexagonal tessellations[3].

It is not necessary to restrict Life to 2 dimensions. Candidate variants have also been discovered in 3 dimensions that are, as Dewdney puts it, "worthy of the name"[5]. These include (but are not limited to) cubic and hexahedral tessellations[1].

Many other attempts at a valid Game of Life using non-regular grids were made online[9]. Some interesting variants include the Penrose rhombii grid, the 3-Isohedral convex pentagon grid, and the House grid. However, insufficient research has been made to find interesting patterns of life here.

III. Proposition

The universe of the proposed Game of Life is an infinite two-dimensional Pythagorean tiling (also known as the two-square tiling)[11], where every smaller square is exactly a quarter of the area of the larger square. Each square is a "cell". Each small square is referred to as a "mini-cell", and each large square is referred to as a "mega-cell". A segment of such a universe is visualized in Figure 5.

A. *Speed of Light*

In this proposition, the Speed of Light (c) is computed similar to how it was in section II-A. The maximum distance that can be covered in one generation is the length of one mega-cell.

B. Neighboring influence

Each cell interacts with its neighbors, which are the cells around it that are adjacent to any point of its border. A mini-cell has 4 mega-cell neighbors and 0 mini-cell neighbors; a mega-cell has 4 mega-cell neighbors and 4 mini-cell neighbors.

This interplay between larger and smaller cells is more akin to the reality of Life - some cells are more influential than others, just as some life forms and structures are more influential than others. In this proposed Game of Life, mini-cells are akin to resources in real life, and mega-cells are akin to neighborhoods in real life.

When evaluating the neighbors of a mega-cell, it is evident that a neighboring mega-cell covers twice as much of the border as compared to the border coverage of a mini-cell. That is, a living mega-cell neighbor is twice as "influential" as a living mini-cell neighbor. (Implicitly, dead neighbors do not influence the cell, irrespective of their size.)

So, the concept of influence points is introduced. Each alive mini-cell neighbor can contribute 1 influence point, while each alive mega-cell neighbor can contribute 2 influence points.

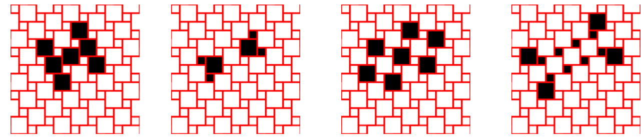
(When comparing this to the original Conway's Game of Life on a Moore lattice, each cell has 8 equally-sized neighbors, and each alive neighbor will contribute 1 influence point.)

C. Cell transitions

After every step in time, the following transitions occur, as described in Algorithm 1.

Algorithm 1: Cell transition rules for the Game of Life on a Pythagorean Tessellation

```
if cell is a mini-cell then
    if cell is alive then
        cell dies
    else
        if cell has exactly 4 neighboring influence
          points then
            cell becomes alive
        end
    end
else
    if cell is alive then
        if cell has less than 5 neighboring influence
          points OR more than 7 neighboring influence
          points then
            cell dies
        end
    else
        if cell has exactly 3, 4, OR 5 neighboring
          influence points then
            cell becomes alive
        end
    end
end
```



(a) (b) (c) (d)

Fig. 6: Oscillator - 'H'

To summarize Algorithm 1, there are similar but different rules followed by mini-cells and mega-cells.

1) Mini-cell transitions
   a) A living mini-cell dies in the next generation. This represents the consumption of a resource in real life.
   b) A dead mini-cell becomes alive in the next generation if it has exactly 4 neighboring influence points. This represents the replenishment of a resource in real life.
2) Mega-cell transitions
   a) A living mega-cell dies in the next generation if it has less than 5 neighboring influence points. This represents death from underpopulation in real life.
   b) A living mega-cell dies in the next generation if it has more than 7 neighboring influence points. This represents death from overpopulation in real life.
   c) A dead mega-cell becomes alive in the next generation if it has exactly 3, 4, or 5 neighboring influence points. This represents the healthy spread of neighborhoods in real life.

IV. Results

With the proposed Game of Life outlined in section III, several life forms can be constructed. Using Harold's categorization[10], the constructions discovered so far can be classified into Oscillators, Gliders, and Guns.

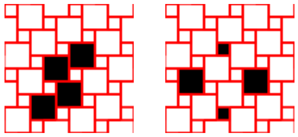
Still life

With this proposed Game of Life, a still life form (ie. an oscillator with period 1) is not possible. Such a hypothetical still-life form cannot have any mini-cells (because by algorithm 1, if a mini-cell is alive, it must die in the next generation, and this implies that the life form is not "still"). Therefore, this still-life form must be comprised of purely mega-cells. For a mega-cell to survive to the next generation, it must have exactly 5, 6, or 7 neighboring influence points. But given that no mini-cells are allowed, the neighboring influence points must be even (because each neighboring mega-cell contributes 2 influence points). Therefore, every cell in this still-life form must have exactly 3 neighboring alive mega-cells. But such a pattern is impossible.

A. Oscillators

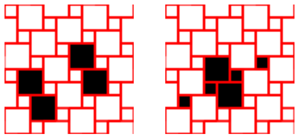
This section illustrates the different states of oscillators discovered so far in subsequent generations.

Figure 6 depicts an oscillator called an 'H' with period 4.



(a) (b)

Fig. 7: Oscillator - ’S’



(a) (b)

Fig. 8: Oscillator - ’Gateway’

Figure 7 depicts an oscillator called an ’S’ with period 2.

Figure 8 depicts an oscillator called a Gateway with period 2.

Figure 9 depicts an oscillator called a Half-Gateway with period 2.

Figure 10 depicts an oscillator called a Ribbon with period 2.

Figure 11 depicts an oscillator called a Lamp with period 2.

Figure 12 depicts an oscillator called a Fish with period 4.

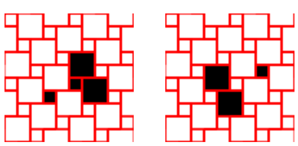
Figure 13 depicts an oscillator called a Tree with period 4.

B. *Gliders*

This section illustrates the different states of gliders discovered so far in subsequent generations. The last subfigure of each figure listed below is identical to the first subfigure’s configuration, but it has translated across space and time.

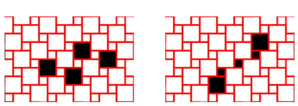
Figure 14 depicts a glider called an Eagle with period 4. The configurations in the original state (fig. 14a) and final state (fig. 14e) are identical, but it has translated across space by a distance of 2 mega-cells. Therefore, its speed is $c/2$.

Figure 15 depicts a glider called a Jellyfish with period 10. Its speed is $c/2$.



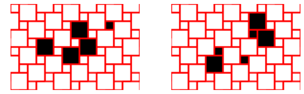
(a) (b)

Fig. 9: Oscillator - ’Half-Gateway’



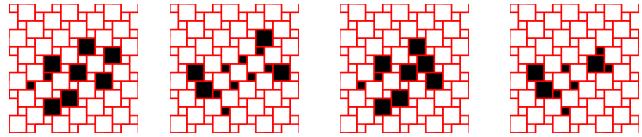
(a) (b)

Fig. 10: Oscillator - ’Ribbon’



(a) (b)

Fig. 11: Oscillator - ’Lamp’



(a) (b) (c) (d)

Fig. 12: Oscillator - ’Fish’

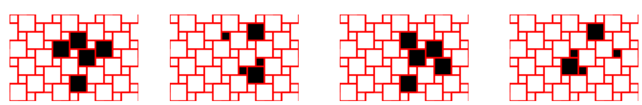
Figure 16 depicts a glider called a Rose with period 10. Its speed is $c/2$.

Figure 17 depicts a glider called a Truck with period 30. Its speed is $c/2$. A zoomed-in illustration of the initial configuration is provided in fig. 17a. All other subfigures illustrate the different states in subsequent generations.

C. *Guns*

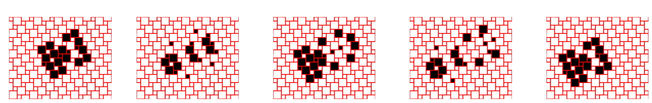
This section illustrates the different states of guns discovered so far in subsequent generations.

Figure 18 depicts a bidirectional gun. It is slightly different from other conventional guns discovered in Conway’s Game of Life, as this pattern indefinitely spawns replicas of its original state, but translating across space facing opposite directions. This is an example of limitless growth[7]. Figure 18a is a zoomed-in image of fig. 18b, which is the initial configuration of the gun. Figure 18c is the state after 22 generations; a replica of the initial configuration is visible 11 mega-tiles to the bottom-right of its original position, along with other ”debris” in its trail. Figure 18d is the state at generation #44, where the initial configuration has translated 22 mega-tiles from its starting configuration, with some other form of debris in its trails. Figure 18e is the state at generation #51, where a replica of the initial configuration is visible 25 mega-tiles away, but translated in the opposite direction and rotated 180 degrees. This pattern repeats itself, and is illustrative of the type ”bidirectional gun”.



(a) (b) (c) (d)

Fig. 13: Oscillator - ’Tree’



(a) (b) (c) (d) (e)

Fig. 14: Glider - ’Eagle’

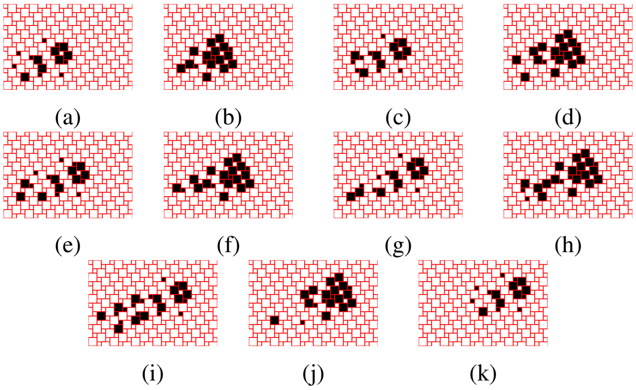


Fig. 15: Glider - 'Jellyfish'

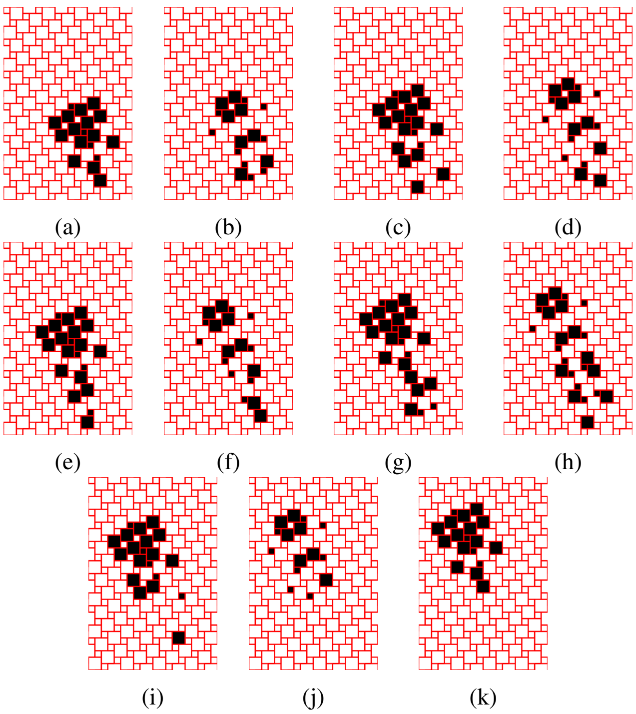


Fig. 16: Glider - 'Rose'

V. Conclusion

The rules specified in algorithm 1 for this proposed Game of Life satisfies the three contraints outlined by Carter Bays[2]. (Admittedly, the first constraint had to be modified to account for two different classes of cells, which is a novelty that Conway's original Game of Life didn't have to consider.)

1) All adjacent mega-cells are treated the same. All adjacent mini-cells are treated the same.
2) At least one glider exists.
3) When starting with a random pattern of cells in a finite wrapped universe, after a finite number of generations, all patterns eventually either disappear, or decompose into one or more oscillators.

Future work in this space includes, but is not limited to:

1) Find a puffer train[10].

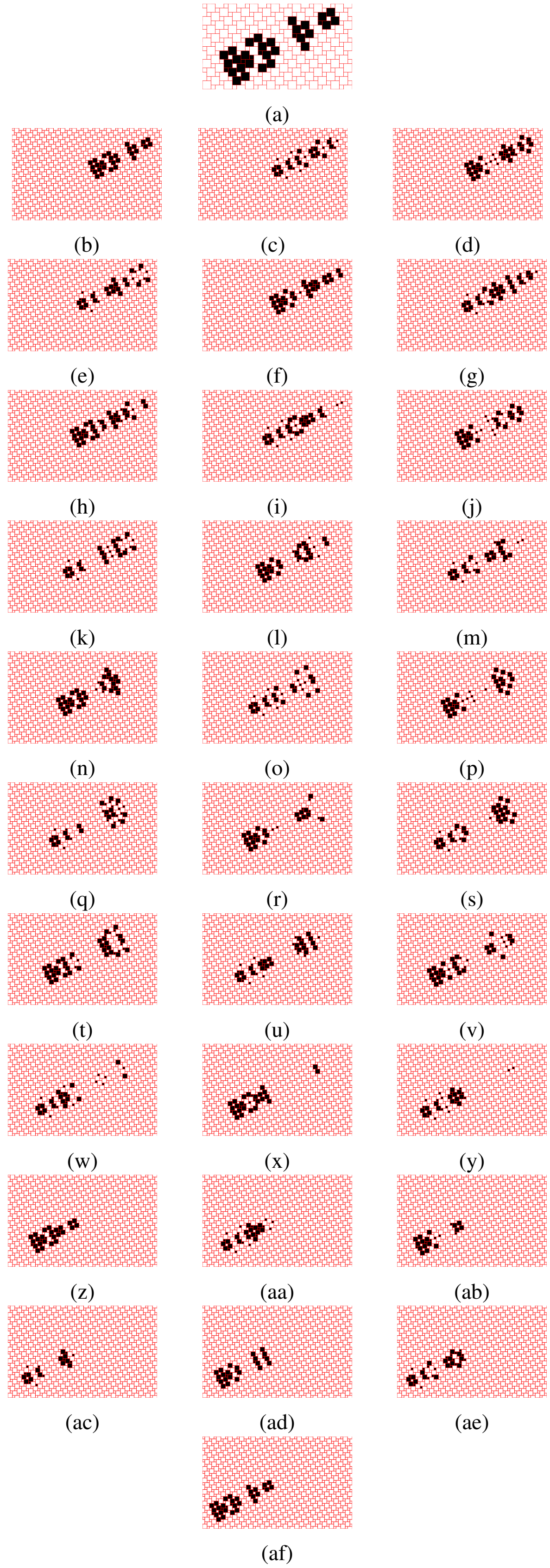
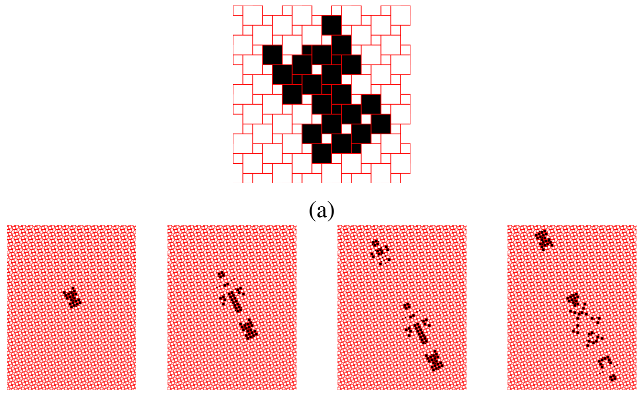


Fig. 17: Glider - 'Truck'



(a)

(b) (c) (d) (e)

Fig. 18: Gun - 'Racecar-Launcher'

2) Find a unidirectional gun.
3) Find constructions that are analogous to a NOT gate, AND gate, and OR gate.
4) Design a universal Turing machine[12] in this Pythagorean tessellation.

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