Short Note on the Riemann Hypothesis

Frank Vega

December 27, 2021

Noname manuscript No.
(will be inserted by the editor)

Short Note on the Riemann Hypothesis

Frank Vega

the date of receipt and acceptance should be inserted later

Abstract Robin criterion states that the Riemann hypothesis is true if and only if the inequality $\sigma(n) < e^{\gamma} \times n \times \log\log n$ holds for all natural numbers $n > 5040$, where $\sigma(n)$ is the sum-of-divisors function of n and $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. Let $q_1 = 2, q_2 = 3, \ldots, q_m$ denote the first m consecutive primes, then an integer of the form $\prod_{i=1}^{m} q_i^{a_i}$ with $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$ is called an Hardy-Ramanujan integer. If the Riemann hypothesis is false, then there are infinitely many Hardy-Ramanujan integers $n > 5040$ such that Robin inequality does not hold and we prove that $n^{\left(1-\frac{0.6253}{\log q_m}\right)} < N_m$, where $N_m = \prod_{i=1}^{m} q_i$ is the primorial number of order m and q_m is the largest prime divisor of n. In addition, we show that q_m will not have an upper bound by some positive value for these counterexamples and therefore, the value of q_m tends to infinity as n goes to infinity.

Keywords Riemann hypothesis · Robin inequality · sum-of-divisors function · prime numbers

Mathematics Subject Classification (2010) MSC 11M26 · MSC 11A41 · MSC 11A25

1 Introduction

In mathematics, the Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$ [7]. Let $N_m = 2 \times 3 \times 5 \times 7 \times 11 \times \cdots \times q_m$ denotes a primorial number of order m such that q_m is the m^{th} prime number [5]. As usual $\sigma(n)$ is the sum-of-divisors function of n [1]:

$$\sum_{d|n} d$$

F. Vega
CopSonic, 1471 Route de Saint-Nauphary 82000 Montauban, France
ORCiD: 0000-0001-8210-4126
E-mail: vega.frank@gmail.com

where $d \mid n$ means the integer d divides n and $d \nmid n$ means the integer d does not divide n. Define $f(n)$ to be $\frac{\sigma(n)}{n}$. Say $\mathsf{Robins}(n)$ holds provided

$$f(n) < e^{\gamma} \times \log\log n.$$

The constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and log is the natural logarithm. The importance of this property is:

Theorem 1.1 $\mathsf{Robins}(n)$ *holds for all natural numbers* $n > 5040$ *if and only if the Riemann hypothesis is true [7]. Moreover, if the Riemann hypothesis is false, then there are infinitely many natural numbers* $n > 5040$ *such that* $\mathsf{Robins}(n)$ *does not hold [7].*

It is known that $\mathsf{Robins}(n)$ holds for many classes of numbers n. $\mathsf{Robins}(n)$ holds for all natural numbers $n > 5040$ that are not divisible by 2 [1]. We recall that an integer n is said to be square free if for every prime divisor q of n we have $q^2 \nmid n$ [1].

Theorem 1.2 $\mathsf{Robins}(n)$ *holds for all natural numbers* $n > 5040$ *that are square free [1].*

Let $q_1 = 2, q_2 = 3, \ldots, q_m$ denote the first m consecutive primes, then an integer of the form $\prod_{i=1}^{m} q_i^{a_i}$ with $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$ is called an Hardy-Ramanujan integer [1]. Based on the theorem 1.1, we know this result:

Theorem 1.3 *If the Riemann hypothesis is false, then there are infinitely many natural numbers* $n > 5040$ *which are an Hardy-Ramanujan integer and* $\mathsf{Robins}(n)$ *does not hold [1].*

We prove if the Riemann hypothesis is false, then there are infinitely many Hardy-Ramanujan integers $n > 5040$ such that $\mathsf{Robins}(n)$ does not hold and $n^{\left(1-\frac{0.6253}{\log q_m}\right)} < N_m$, where $N_m = \prod_{i=1}^{m} q_i$ is the primorial number of order m and q_m is the largest prime divisor of n. Furthermore, we show that q_m will not have an upper bound by some positive value for these counterexamples and thus, the value of q_m tends to infinity as n goes to infinity.

2 Known Results

These are known results:

Theorem 2.1 *[1]. For* $n > 1$*:*

$$f(n) < \prod_{q \mid n} \frac{q}{q-1}.$$

Theorem 2.2 *[2].*

$$\prod_{k=1}^{\infty} \frac{1}{1-\frac{1}{q_k^2}} = \zeta(2) = \frac{\pi^2}{6}.$$

Theorem 2.3 *[3]. Let $n > e^{e^{23.762143}}$ and let all its prime divisors be $q_1 < \cdots < q_m$, then*

$$\left(\prod_{i=1}^{m} \frac{q_i}{q_i - 1}\right) < \frac{1771561}{1771560} \times e^{\gamma} \times \log\log n.$$

Theorem 2.4 $\mathsf{Robins}(n)$ *holds for all natural numbers $10^{10^{13.11485}} \geq n > 5040$ [6].*

Theorem 2.5 *[9]. For $q_m \geq 20000$, we have*

$$\log q_m < \log\log N_m + \frac{0.1253}{\log q_m}.$$

Theorem 2.6 *[8]. For $x \geq 286$:*

$$\prod_{q \leq x} \frac{q}{q-1} < e^{\gamma} \times (\log x + \frac{1}{2 \times \log(x)}).$$

Theorem 2.7 *[4]. For $x > -1$:*

$$\frac{x}{x+1} \leq \log(1+x).$$

3 A Central Theorem

The following is a key theorem. It gives an upper bound on $f(n)$ that holds for all natural numbers n. The bound is too weak to prove $\mathsf{Robins}(n)$ directly, but is critical because it holds for all natural numbers n. Further the bound only uses the primes that divide n and not how many times they divide n.

Theorem 3.1 *Let $n > 1$ and let all its prime divisors be $q_1 < \cdots < q_m$. Then,*

$$f(n) < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}.$$

Proof Putting together the theorems 2.1 and 2.2 yields the proof:

$$f(n) < \prod_{i=1}^{m} \left(\frac{q_i}{q_i - 1}\right) = \prod_{i=1}^{m} \left(\frac{q_i + 1}{q_i} \times \frac{1}{1 - \frac{1}{q_i^2}}\right) < \frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i}.$$

4 A Particular Case

We can easily prove that $\mathsf{Robins}(n)$ is true for certain kind of numbers.

Theorem 4.1 $\mathsf{Robins}(n)$ *holds for $n > 5040$ when $q \leq 5$, where q is the largest prime divisor of n.*

Proof Let $n > 5040$ and let all its prime divisors be $q_1 < \cdots < q_m \leq 5$, then we need to prove

$$f(n) < e^{\gamma} \times \log\log n$$

that is true when

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq e^{\gamma} \times \log\log n$$

according to the theorem 2.1. For $q_1 < \cdots < q_m \leq 5$,

$$\prod_{i=1}^{m} \frac{q_i}{q_i - 1} \leq \frac{2 \times 3 \times 5}{1 \times 2 \times 4} = 3.75 < e^{\gamma} \times \log\log(5040) \approx 3.81.$$

However, we know for $n > 5040$

$$e^{\gamma} \times \log\log(5040) < e^{\gamma} \times \log\log n$$

and therefore, the proof is complete when $q_1 < \cdots < q_m \leq 5$.

5 Robin on Divisibility

The next theorem implies that $\mathsf{Robins}(n)$ holds for a wide range of natural numbers $n > 5040$.

Theorem 5.1 $\mathsf{Robins}(n)$ *holds for all natural numbers* $n > 5040$ *when a prime* $q \leq 1771559$ *complies with* $q \nmid n$*.*

Proof Note that $f(n) < \frac{n}{\varphi(n)} = \prod_{q \mid n} \frac{q}{q-1}$ from the theorem 2.1, where $\varphi(x)$ is the Euler's totient function. We have that $f(n) < \frac{1771561}{1771560} \times e^{\gamma} \times \log\log(n)$ for any number $n > 10^{10^{13.11485}}$. Suppose that n is not divisible by a prime q for q less than or equal to some prime bound Q and $n > N = 10^{10^{13.11485}}$. Then,

$$\begin{aligned} f(n) &< \frac{n}{\varphi(n)} \\ &= \frac{n \times q}{\varphi(n \times q)} \times \frac{q-1}{q} \\ &< \frac{1771561}{1771560} \times \frac{q-1}{q} \times e^{\gamma} \times \log\log(n \times q) \end{aligned}$$

and

$$\begin{aligned}\frac{f(n)}{e^{\gamma}\times\log\log(n)} &< \frac{1771561}{1771560}\times\frac{q-1}{q}\times\frac{\log\log(n\times q)}{\log\log(n)}\\ &\leq \frac{1771561}{1771560}\times\frac{Q-1}{Q}\times\frac{\log\log(n\times Q)}{\log\log(n)}\\ &= \frac{1771561}{1771560}\times\frac{Q-1}{Q}\times\frac{\log\log(n)+\log(1+\frac{\log(Q)}{\log(n)})}{\log\log(n)}\\ &= \frac{1771561}{1771560}\times\frac{Q-1}{Q}\times\left(1+\frac{\log(1+\frac{\log(Q)}{\log(n)})}{\log\log(n)}\right)\end{aligned}$$

So

$$\frac{f(n)}{e^{\gamma}\times\log\log(n)} < \frac{1771561}{1771560}\times\frac{Q-1}{Q}\times\left(1+\frac{\log(1+\frac{\log(Q)}{\log(n)})}{\log\log(n)}\right)$$

for $n > N = 10^{10^{13.11485}}$. The right hand side is less than 1 for $Q \leq 1771559$. Moreover, note that the inequality $10^{10^{13.11485}} > e^{e^{23.762143}}$ is satisfied. Therefore, $\mathsf{Robins}(n)$ holds as a consequence of the theorems 2.3 and 2.4.

6 A Main Insight

The next theorem is a main insight.

Theorem 6.1 *Let* $\frac{\pi^2}{6}\times\log\log n' \leq \log\log n$ *for some natural number* $n > 5040$ *such that* n' *is the square free kernel of the natural number* n*. Then* $\mathsf{Robins}(n)$ *holds.*

Proof Let n' be the square free kernel of the natural number n, that is the product of the distinct primes $q_1,\ldots,q_m$. By assumption we have that

$$\frac{\pi^2}{6}\times\log\log n' \leq \log\log n.$$

For all square free $n' \leq 5040$, $\mathsf{Robins}(n')$ holds if and only if $n' \notin \{2,3,5,6,10,30\}$ [1]. $\mathsf{Robins}(n)$ holds for all natural numbers $n > 5040$ when $n' \in \{2,3,5,6,10,15,30\}$ due to the theorem 4.1. When $n' > 5040$, we know that $\mathsf{Robins}(n')$ holds and so

$$f(n') < e^{\gamma}\times\log\log n'$$

because of the theorem 1.2. By the previous theorem 3.1:

$$f(n) < \frac{\pi^2}{6}\times\prod_{i=1}^{m}\frac{q_i+1}{q_i}.$$

Suppose by way of contradiction that $\mathsf{Robins}(n)$ fails. Then

$$f(n) \geq e^{\gamma} \times \log \log n.$$

We claim that

$$\frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i} > e^{\gamma} \times \log \log n.$$

Since otherwise we would have a contradiction. This shows that

$$\frac{\pi^2}{6} \times \prod_{i=1}^{m} \frac{q_i + 1}{q_i} > \frac{\pi^2}{6} \times e^{\gamma} \times \log \log n'.$$

Thus

$$\prod_{i=1}^{m} \frac{q_i + 1}{q_i} > e^{\gamma} \times \log \log n',$$

and

$$\prod_{i=1}^{m} \frac{q_i + 1}{q_i} > f(n'),$$

This is a contradiction since $f(n')$ is equal to

$$\frac{(q_1 + 1) \times \cdots \times (q_m + 1)}{q_1 \times \cdots \times q_m}$$

according to the formula $f(x)$ for the square free numbers [1].

7 Proof of Main Theorem

Theorem 7.1 *If the Riemann hypothesis is false, then there are infinitely many Hardy-Ramanujan integers $n > 5040$ such that $\mathsf{Robins}(n)$ does not hold and $n^{\left(1 - \frac{0.6253}{\log q_m}\right)} < N_m$, where $N_m = \prod_{i=1}^{m} q_i$ is the primorial number of order m and q_m is the largest prime divisor of n. In addition, q_m will not have an upper bound by some positive value for these counterexamples and therefore, the value of q_m tends to infinity as n goes to infinity.*

Proof Let $\prod_{i=1}^{m} q_i^{a_i}$ be the representation of some natural number $n > 5040$ as a product of primes $q_1 < \cdots < q_m$ with natural numbers as exponents $a_1, \ldots, a_m$. The primes $q_1 < \cdots < q_m$ must be the first m consecutive primes and $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$ since the natural number $n > 5040$ will be an Hardy-Ramanujan integer. We assume that $\mathsf{Robins}(n)$ does not hold. Indeed, we know there are infinitely many Hardy-Ramanujan integers such as $n > 5040$ when the Riemann hypothesis is false according to the theorem 1.3. From the theorem 5.1, we know that necessarily $q_m \geq 1771559$. So,

$$e^{\gamma} \times \log \log n \leq f(n) < \prod_{q \leq q_m} \frac{q}{q-1} < e^{\gamma} \times (\log q_m + \frac{1}{2 \times \log(q_m)})$$

because of the theorems 2.1 and 2.6. Hence,

$$\log\log n < \log q_m + \frac{0.5}{\log(q_m)}.$$

From the theorem 2.5, we have that

$$\log\log n < \log\log N_m + \frac{0.1253}{\log q_m} + \frac{0.5}{\log(q_m)}.$$

That is the same as

$$\log\log n - \log\log N_m < \frac{0.6253}{\log q_m}.$$

Then,

$$\begin{aligned}\log\log n - \log\log N_m &= \log\left(\log N_m + \log(\frac{n}{N_m})\right) - \log\log N_m \\ &= \log\left(\log N_m \times (1 + \frac{\log(\frac{n}{N_m})}{\log N_m})\right) - \log\log N_m \\ &= \log\log N_m + \log(1 + \frac{\log(\frac{n}{N_m})}{\log N_m}) - \log\log N_m \\ &= \log(1 + \frac{\log(\frac{n}{N_m})}{\log N_m}).\end{aligned}$$

In addition, we know that

$$\log(1 + \frac{\log(\frac{n}{N_m})}{\log N_m}) \geq \frac{\log(\frac{n}{N_m})}{\log n}$$

using the theorem 2.7 since $\frac{\log(\frac{n}{N_m})}{\log N_m} > -1$. Certainly, we will have that

$$\log(1 + \frac{\log(\frac{n}{N_m})}{\log N_m}) \geq \frac{\frac{\log(\frac{n}{N_m})}{\log N_m}}{\frac{\log(\frac{n}{N_m})}{\log N_m} + 1} = \frac{\log(\frac{n}{N_m})}{\log(\frac{n}{N_m}) + \log N_m} = \frac{\log(\frac{n}{N_m})}{\log n}.$$

In this way, we have that

$$\frac{\log(\frac{n}{N_m})}{\log n} < \frac{0.6253}{\log q_m}$$

which is equivalent to

$$\log(\frac{n}{N_m}) < \log(n^{\frac{0.6253}{\log q_m}})$$

and thus

$$\frac{n}{N_m} < n^{\frac{0.6253}{\log q_m}}.$$

Finally, we obtain that

$$n^{\left(1 - \frac{0.6253}{\log q_m}\right)} < N_m.$$

Moreover, we know that q_m will not have an upper bound by some positive value for these counterexamples because of the theorem 6.1. Certainly, if there is a possible upper bound for q_m, then it cannot exist infinitely many Hardy-Ramanujan integers $n > 5040$ such that $\mathsf{Robins}(n)$ does not hold as a consequence of the theorem 6.1.

Acknowledgments

The author would like to thank his mother, maternal brother and his friend Sonia for their support.

References

1. Choie, Y., Lichiardopol, N., Moree, P., Solé, P.: On Robin's criterion for the Riemann hypothesis. Journal de Théorie des Nombres de Bordeaux **19**(2), 357–372 (2007). DOI doi:10.5802/jtnb.591
2. Edwards, H.M.: Riemann's Zeta Function. Dover Publications (2001)
3. Hertlein, A.: Robin's Inequality for New Families of Integers. Integers **18** (2018)
4. Kozma, L.: Useful Inequalities. `http://www.lkozma.net/inequalities_cheat_sheet/ineq.pdf` (2021). Accessed on 2021-12-27
5. Nicolas, J.L.: Petites valeurs de la fonction d'Euler. Journal of number theory **17**(3), 375–388 (1983). DOI 10.1016/0022-314X(83)90055-0
6. Platt, D.J., Morrill, T.: Robin's inequality for 20-free integers. INTEGERS: Electronic Journal of Combinatorial Number Theory (2021)
7. Robin, G.: Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann. J. Math. pures appl **63**(2), 187–213 (1984)
8. Rosser, J.B., Schoenfeld, L.: Approximate Formulas for Some Functions of Prime Numbers. Illinois Journal of Mathematics **6**(1), 64–94 (1962). DOI doi:10.1215/ijm/1255631807
9. Solé, P., Planat, M.: Robin inequality for $7-$ free integers. Integers: Electronic Journal of Combinatorial Number Theory **11**, A65 (2011)