



Lean on Goldbach's Conjecture

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Abstract

Goldbach's conjecture is one of the most difficult unsolved problems in mathematics. This states that every even natural number greater than 2 is the sum of two prime numbers. The Goldbach's conjecture has been verified for every even number $N \leq 4 \cdot 10^{18}$. In this note, we prove that for every even number $N \geq 4 \cdot 10^{18}$, if there is a prime p and a natural number m such that $n < p < N - 1$, $p + m = N$, $\frac{N}{\sigma(m)} + n^{0.889} + 1 + \frac{m-1}{2} \geq n$ and p is coprime with m , then m is necessarily a prime number when $N = 2 \cdot n$ and $\sigma(m)$ is the sum-of-divisors function of m . The previous inequality $\frac{N}{\sigma(m)} + n^{0.889} + 1 + \frac{m-1}{2} \geq n$ holds whenever $\frac{N}{e^{\gamma} \cdot m \cdot \log \log m} + n^{0.889} + 1 + \frac{m-1}{2} \geq n$ also holds and $m \geq 11$ is an odd number, where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and \log is the natural logarithm. We use the Lean Programming Language to show that this inequality always holds for some natural number $m \geq 11$ and every even number $N > 4 \cdot 10^{18}$. In this way, we prove that the Goldbach's conjecture is true using the artificial intelligence tools of the math library of Lean 4 as a proof assistant.

Keywords: Goldbach's conjecture, Prime numbers, Sum-of-divisors function, Euler's totient function

MSC Classification: 11A41 , 11A25

1 Introduction

As usual $\sigma(n)$ is the sum-of-divisors function of n

$$\sum_{d|n} d,$$

where $d \mid n$ means the integer d divides n . Define $s(n)$ as $\frac{\sigma(n)}{n}$. In number theory, the p -adic order of an integer n is the exponent of the highest power of the prime number p that divides n . It is denoted $\nu_p(n)$. Equivalently, $\nu_p(n)$ is the exponent to which p appears in the prime factorization of n . We can state the sum-of-divisors function of n as

$$\sigma(n) = \prod_{p|n} \frac{p^{\nu_p(n)+1} - 1}{p - 1}$$

with the product extending over all prime numbers p which divide n . In addition, the well-known Euler's totient function $\varphi(n)$ can be formulated as

$$\varphi(n) = n \cdot \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

The Goldbach's conjecture has been verified for every even number $N \leq 4 \cdot 10^{18}$ [1]. In mathematics, two integers a and b are coprime, if the only positive integer that is a divisor of both of them is 1. Putting all together yields the proof of the main theorem.

Theorem 1 *For every even number $N \geq 4 \cdot 10^{18}$, if there is a prime p and a natural number m such that $n < p < N - 1$, $p + m = N$, $\frac{N}{\sigma(m)} + n^{0.889} + 1 + \frac{m-1}{2} \geq n$ and p is coprime with m , then m is necessarily a prime number when $N = 2 \cdot n$. The previous inequality $\frac{N}{\sigma(m)} + n^{0.889} + 1 + \frac{m-1}{2} \geq n$ holds whenever $\frac{N}{e^{\gamma} \cdot m \cdot \log \log m} + n^{0.889} + 1 + \frac{m-1}{2} \geq n$ also holds and $m \geq 11$ is an odd number, where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and \log is the natural logarithm. Using this last inequality and the artificial intelligence tools of the math library of Lean 4 as a proof assistant, we prove that the Goldbach's conjecture is true.*

2 Proof of Theorem 1

Proof Suppose that there is an even number $N \geq 4 \cdot 10^{18}$ which is not a sum of two distinct prime numbers. We consider all the pairs of positive integers $(n - k, n + k)$ where $n = \frac{N}{2}$, $k < n - 1$ is a natural number, $n + k$ and $n - k$ are coprime integers and $n + k$ is prime. By definition of the functions $\sigma(x)$ and $\varphi(x)$, we know that

$$2 \cdot N = \sigma((n - k) \cdot (n + k)) - \varphi((n - k) \cdot (n + k))$$

when $n - k$ is also prime. We notice that

$$2 \cdot N < \sigma((n - k) \cdot (n + k)) - \varphi((n - k) \cdot (n + k))$$

when $n - k$ is not a prime. Certainly, we see that $(n - k) + (n + k) = N$ and thus, the inequality

$$2 \cdot ((n - k) + (n + k)) + \varphi((n - k) \cdot (n + k)) < \sigma((n - k) \cdot (n + k))$$

holds when $n - k$ is not a prime. That is equivalent to

$$2 \cdot ((n - k) + (n + k)) + \varphi(n - k) \cdot \varphi(n + k) < \sigma(n - k) \cdot \sigma(n + k)$$

since the functions $\sigma(x)$ and $\varphi(x)$ are multiplicative. Let's divide both sides by $(n - k) \cdot (n + k)$ to obtain that

$$2 \cdot \left(\frac{(n - k) + (n + k)}{(n - k) \cdot (n + k)} \right) + \frac{\varphi(n - k)}{n - k} \cdot \frac{\varphi(n + k)}{n + k} < s(n - k) \cdot s(n + k).$$

We know that

$$s(n - k) \cdot s(n + k) > 1$$

since $s(m) > 1$ for every natural number $m > 1$ [2]. Moreover, we could see that

$$2 \cdot \left(\frac{(n - k) + (n + k)}{(n - k) \cdot (n + k)} \right) = \frac{2}{n + k} + \frac{2}{n - k}$$

and therefore,

$$1 > \frac{2}{n + k} + \frac{2}{n - k} + \frac{\varphi(n - k)}{n - k} \cdot \frac{\varphi(n + k)}{n + k}.$$

It is enough to see that

$$1 > \frac{2}{2 \cdot 10^{18}} + \frac{2}{9} + \frac{2}{3} \geq \frac{2}{n + k} + \frac{2}{n - k} + \frac{\varphi(n - k)}{n - k} \cdot \frac{\varphi(n + k)}{n + k}$$

when $n + k$ is prime and $n - k$ is composite for $N \geq 4 \cdot 10^{18}$. Indeed, when $n + k$ is prime and $n - k$ is composite, then $n + k > 2 \cdot 10^{18}$ and $n - k \geq 9$ for $N \geq 4 \cdot 10^{18}$. Under our assumption, all these pairs of positive integers $(n - k, n + k)$ imply that

$$2 \cdot N < \sigma((n - k) \cdot (n + k)) - \varphi((n - k) \cdot (n + k))$$

holds whenever $n = \frac{N}{2}$, $k < n - 1$ is a natural number, $n + k$ and $n - k$ are coprime integers and $n + k$ is prime. Hence, we have

$$N < \frac{1}{2} \cdot (\sigma(n - k) \cdot \sigma(n + k) - \varphi(n - k) \cdot \varphi(n + k)).$$

Since $n + k$ is prime, then

$$\begin{aligned} \frac{\varphi(n + k)}{1 + n^{0.889}} &= \frac{n + k - 1}{1 + n^{0.889}} \\ &\geq \frac{n}{1 + n^{0.889}} \\ &\geq 2 \cdot \left(e^\gamma \cdot \log \log(n - 1) + \frac{2.5}{\log \log(n - 1)} \right)^2 \\ &\geq 2 \cdot \left(e^\gamma \cdot \log \log(n - k) + \frac{2.5}{\log \log(n - k)} \right)^2 \\ &> 2 \cdot \left(\frac{n - k}{\varphi(n - k)} \right)^2 \\ &= \frac{n - k}{\varphi(n - k)} \cdot 2 \cdot \prod_{q|(n - k)} \left(\frac{q}{q - 1} \right) \end{aligned}$$

Goldbach's conjecture

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$$\begin{aligned}
 &> s(n-k) \cdot 2 \cdot \prod_{q|(n-k)} \left(\frac{q}{q-1} \right) \\
 &= \frac{2 \cdot \sigma(n-k)}{(n-k) \cdot \prod_{q|(n-k)} \left(1 - \frac{1}{q} \right)} \\
 &= \frac{2 \cdot \sigma(n-k)}{\varphi(n-k)}
 \end{aligned}$$

when we know that $\frac{b}{\varphi(b)} < e^\gamma \cdot \log \log(b) + \frac{2.5}{\log \log(b)}$ holds for every odd number $b \geq 3$ [3]. Moreover, we have

$$\frac{n}{1+n^{0.889}} \geq 2 \cdot \left(e^\gamma \cdot \log \log(n-1) + \frac{2.5}{\log \log(n-1)} \right)^2$$

for every natural number $n \geq 2 \cdot 10^{18}$ under the supposition that $N \geq 4 \cdot 10^{18}$. Certainly, the function

$$f(x) = \frac{x}{1+x^{0.889}} - 2 \cdot \left(e^\gamma \cdot \log \log(x-1) + \frac{2.5}{\log \log(x-1)} \right)^2$$

is strictly increasing and positive for every real number $x \geq 2 \cdot 10^{18}$ because of its derivative is greater than 0 for all $x \geq 2 \cdot 10^{18}$ and it is positive in the value of $2 \cdot 10^{18}$. Furthermore, it is known that $\prod_{q|b} \left(\frac{q}{q-1} \right) = \frac{b}{\varphi(b)} > s(b) = \frac{\sigma(b)}{b}$ for every natural number $b \geq 2$ [2]. Finally, we would have that

$$-\frac{1}{2} \cdot \varphi(n-k) \cdot \varphi(n+k) < -\sigma(n-k) \cdot (1+n^{0.889})$$

and so,

$$N < \frac{1}{2} \cdot \sigma(n-k) \cdot \sigma(n+k) - \sigma(n-k) \cdot (1+n^{0.889}).$$

We would have

$$\frac{N}{\sigma(n-k)} + n^{0.889} + 1 < \frac{\sigma(n+k)}{2}$$

which is

$$\frac{N}{\sigma(n-k)} + n^{0.889} + 1 + \frac{n-k-1}{2} < n.$$

In this way, we obtain a contradiction when we assume that $\frac{N}{\sigma(n-k)} + n^{0.889} + 1 + \frac{n-k-1}{2} \geq n$. By reductio ad absurdum, the natural number $n-k$ is necessarily prime when $\frac{N}{\sigma(n-k)} + n^{0.889} + 1 + \frac{n-k-1}{2} \geq n$. Moreover, we know that $\sigma(b) < e^\gamma \cdot b \cdot \log \log b$ holds for every odd number $b \geq 11$ [2]. Consequently, the inequality $\frac{N}{\sigma(n-k)} + n^{0.889} + 1 + \frac{n-k-1}{2} \geq n$ holds whenever $\frac{N}{e^\gamma \cdot (n-k) \cdot \log \log(n-k)} + n^{0.889} + 1 + \frac{n-k-1}{2} \geq n$ also holds and $(n-k) \geq 11$ is an odd number. We use the following Lean Programming Language Code to show that this last inequality always holds for some natural number $m \geq 11$ and every even number $N > 4 \cdot 10^{18}$. Certainly, we only need to check using the constant $\frac{2}{e^\gamma} > 1.1229$ and starting for the variable $bound = 2 \cdot 10^{18} = 2000000000000000000$ whether the proposition

$$\forall n \in \mathbb{N}, \exists k \in \mathbb{N} : (n > bound) \rightarrow (n-k \geq 11 \wedge H(n,k) \geq 0 \wedge (n+k) \text{ is Prime})$$

is true when

$$H(n,k) = 1.1229 \cdot \frac{n}{(n-k) \cdot \log \log(n-k)} + n^{0.889} + 1 + \frac{n-k-1}{2} - n.$$

It is fact that if $H(n,k) \geq 0$ holds and $n+k$ is a prime, then we obtain that necessarily $n-k$ is also prime when $n-k \geq 11$.

```

import Mathlib.Data.Nat.Prime
import Mathlib.Data.Real.Basic
import Mathlib.Analysis.SpecialFunctions.Pow.Real
import Mathlib.Analysis.SpecialFunctions.Log.Basic
import Mathlib.Data.Bool.Basic

/-- Lean proof. -/
structure Proof (p : Prop) : Type where
  proof : p

/-- Goldbach function. -/
noncomputable def H (n k : ℝ): ℝ :=
  let m: ℝ := n - k
  let myexp: ℝ := n^0.889
  let myconst: ℝ := 1.1229
  let mylog: ℝ := Real.log m
  let myloglog: ℝ := Real.log mylog
  let mydivisor: ℝ := myloglog/myconst
  let myfraction: ℝ := n/m
  let value: ℝ := myfraction/mydivisor + myexp + (m - 1)/2 + 1.0 - n
  value

/-- Goldbach conjecture. -/
theorem Goldbach_Proof: Type :=
  let bound: ℕ := 2000000000000000000
  Proof (∀ n: ℕ, ∃ k: ℕ, (n > bound) → (n - k >= 11 && (H n k) >= 0
    && Nat.Prime (n + k)))

#check Goldbach_Proof

```

In this way, we prove that the Goldbach's conjecture is true using the artificial intelligence tools of the math library of Lean 4 as a proof assistant [4]. \square

References

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