



Note on the Odd Perfect Numbers

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Abstract

The Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. Under the assumption of the Riemann Hypothesis, we prove that there is not any odd perfect number at all.

Keywords: Riemann Hypothesis, Prime numbers, Odd perfect numbers, Superabundant numbers, Sum-of-divisors function

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1. Introduction

The Riemann Hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. As usual $\sigma(n)$ is the sum-of-divisors function of n :

$$\sum_{d|n} d$$

where $d | n$ means the integer d divides n , $d \nmid n$ means the integer d does not divide n and $d^k \parallel n$ means $d^k | n$ and $d^{k+1} \nmid n$. Define $f(n)$ and $G(n)$ to be $\frac{\sigma(n)}{n}$ and $\frac{f(n)}{\log \log n}$ respectively, such that \log is the natural logarithm. We know these properties from these functions:

Proposition 1.1. [1]. Let $\prod_{i=1}^r q_i^{a_i}$ be the representation of n as a product of primes $q_1 < \dots < q_r$ with natural numbers as exponents a_1, \dots, a_r . Then,

$$f(n) = \left(\prod_{i=1}^r \frac{q_i}{q_i - 1} \right) \times \prod_{i=1}^r \left(1 - \frac{1}{q_i^{a_i+1}} \right).$$

Proposition 1.2. For every prime power q^a , we have that $f(q^a) = \frac{q^{a+1}-1}{q^a \times (q-1)}$ [2]. If $m, n \geq 2$ are natural numbers, then $f(m \times n) \leq f(m) \times f(n)$ [2]. Moreover, if p is a prime number, and a, b two positive integers, then [2]:

$$f(p^{a+b}) - f(p^a) \times f(p^b) = -\frac{(p^a - 1) \times (p^b - 1)}{p^{a+b-1} \times (p - 1)^2}.$$

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Say Robins(n) holds provided

$$G(n) < e^\gamma$$

where the constant $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. The importance of this property is:

Proposition 1.3. Robins(n) holds for all natural numbers $n > 5040$ if and only if the Riemann Hypothesis is true [3].

The Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{p \leq x} \log p$$

with the sum extending over all prime numbers p that are less than or equal to x [4]. We state the following property about this function:

Proposition 1.4. [4]. For $x \geq 89909$:

$$\theta(x) > \left(1 - \frac{0.068}{\log(x)}\right) \times x.$$

In mathematics, $\Psi = n \times \prod_{q|n} \left(1 + \frac{1}{q}\right)$ is called the Dedekind Ψ function. Say Dedekind(q_n) holds provided

$$\prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > \frac{e^\gamma}{\zeta(2)} \times \log \theta(q_n)$$

where q_n is the n th prime number, $\zeta(x)$ is the Riemann zeta function and $\zeta(2) = \prod_{i=1}^{\infty} \frac{q_i^2}{q_i^2 - 1} = \frac{\pi^2}{6}$. The importance of this inequality is:

Proposition 1.5. Dedekind(q_n) holds for all prime numbers $q_n > 3$ if and only if the Riemann Hypothesis is true [5].

Let $q_1 = 2, q_2 = 3, \dots, q_k$ denote the first k consecutive primes, then an integer of the form $\prod_{i=1}^k q_i^{a_i}$ with $a_1 \geq a_2 \geq \dots \geq a_k \geq 0$ is called an Hardy-Ramanujan integer [6]. A natural number n is called superabundant precisely when, for all natural numbers $m < n$

$$f(m) < f(n).$$

Proposition 1.6. If n is superabundant, then n is an Hardy-Ramanujan integer [7]. Let n be a superabundant number, then $p \parallel n$ where p is the largest prime factor of n [7]. For large enough superabundant number n , we have that $q^{a_q} < 2^{a_2}$ for $q > 11$ where $q^{a_q} \parallel n$ and $2^{a_2} \parallel n$ [7]. For large enough superabundant number n , we obtain that $\log n < \left(1 + \frac{0.5}{\log p}\right) \times p$ where p is the largest prime factor of n [4]. Let n be a superabundant number, then $f(n) > (1 - \varepsilon(p)) \times \prod_{q|n} \frac{q}{q-1}$ where $\varepsilon(p) = \frac{1}{\log p} \times \left(1 + \frac{1.5}{\log p}\right)$ and p is the largest prime factor of n [4].

Define $R(N_k)$ to be $\frac{f(N_k)}{\log \log N_k}$ where $N_k = \prod_{i=1}^k q_i$ is the primorial number of order k .

Proposition 1.7. [5], [8]:

$$\lim_{k \rightarrow +\infty} \frac{R(N_k)}{2} = \frac{e^\gamma}{\zeta(2)}.$$

Proposition 1.8. [9]. Under the assumption of the Riemann Hypothesis, the inequality $R(N_k) > R(N_{k+1})$ is violated for infinitely many k 's.

In addition, we will use this property:

Proposition 1.9. [5], [6]. For $n \geq 2$:

$$\prod_{q > q_n} \frac{q^2}{q^2 - 1} \leq e^{\frac{2}{q_n}}.$$

In number theory, a perfect number is a positive integer n such that $f(n) = 2$. Euclid proved that every even perfect number is of the form $2^{s-1} \times (2^s - 1)$ whenever $2^s - 1$ is prime. It is unknown whether any odd perfect numbers exist, though various results have been obtained:

Proposition 1.10. Any odd perfect number N must satisfy the following conditions: $N > 10^{1500}$ and the largest prime factor of N is greater than 10^8 [10], [11].

Under the assumption of the Riemann Hypothesis, we prove that there is not any odd perfect number at all.

2. Main Insight

Lemma 2.1. Under the assumption of the Riemann Hypothesis, we prove that

$$\frac{\pi^2}{6.4} \times \prod_{q \leq q_n} \left(1 + \frac{1}{q}\right) > e^\gamma \times \log \theta(q_n)$$

is satisfied for infinitely many prime numbers q_n .

Proof. We know that:

$$\lim_{k \rightarrow +\infty} R(N_k) = \lim_{k \rightarrow +\infty} R(N_k) = \frac{e^\gamma}{\frac{\pi^2}{6}}$$

according to the Proposition 1.7 and the properties of limit [8]. For any positive real number ε , there exists a natural number m such that

$$R(N_k) > \frac{e^\gamma}{\frac{\pi^2}{6}} - \varepsilon$$

for all $k > m$, because of the definition of limit inferior [8]. If we multiply the previous inequality by $\frac{6.4}{6}$, then we obtain that

$$\frac{6.4}{6} \times R(N_k) > \frac{e^\gamma}{\frac{\pi^2}{6.4}} - \varepsilon'$$

no matter how large or small could be the value of ε' . However, there are infinitely many prime numbers q_k such that there always exists some prime q_n :

$$R(N_n) \geq \frac{6.4}{6} \times R(N_k) + \varepsilon'$$

under the assumption of the Riemann Hypothesis due to the Proposition 1.8. Certainly, there could be many primes q_n such that

$$R(N_n) \gg \frac{6.4}{6} \times R(N_k)$$

for every large enough and fixed prime number q_k where $q_n > q_k$ and

$$6 \times \prod_{q_k < q \leq q_n} \left(1 + \frac{1}{q}\right) \gg 6.4 \times \left(1 + \frac{\log \frac{N_n}{N_k}}{\log N_k \times \log \log N_k}\right).$$

Here, the symbol \gg means “much greater than”. Indeed,

$$\begin{aligned} \frac{\log \log N_n}{\log \log N_k} &= \frac{\log \left(\log N_k + \log \frac{N_n}{N_k}\right)}{\log \log N_k} \\ &= \frac{\log \left(\log N_k \times \left(1 + \frac{\log \frac{N_n}{N_k}}{\log N_k}\right)\right)}{\log \log N_k} \\ &= \frac{\log \log N_k + \log \left(1 + \frac{\log \frac{N_n}{N_k}}{\log N_k}\right)}{\log \log N_k} \\ &= 1 + \frac{\log \left(1 + \frac{\log \frac{N_n}{N_k}}{\log N_k}\right)}{\log \log N_k} \\ &\leq 1 + \frac{\log \frac{N_n}{N_k}}{\log N_k \times \log \log N_k} \end{aligned}$$

since when $x > -1$, then $x \geq \log(1 + x)$ [12]. Therefore, the proof is done. \square

3. Main Theorem

Theorem 3.1. *Under the assumption of the Riemann Hypothesis, we prove that there is not any odd perfect number at all.*

Proof. Suppose that N is the smallest odd perfect number, then we will show its existence implies that the Riemann Hypothesis is false. There is always a large enough superabundant number n such that n is a multiple of N . We would have

$$f(n) \leq f(N) \times f\left(\frac{n}{N}\right)$$

according to the Proposition 1.2. That is the same as

$$f(n) \leq 2 \times f\left(\frac{n}{N}\right)$$

since $f(N) = 2$, because N is a perfect number. Hence,

$$\begin{aligned}\frac{f(n)}{2} &= \frac{(2 - \frac{1}{2^{a_2}}) \times f(\frac{n}{2^{a_2}})}{2} \\ &= f(\frac{n}{2^{a_2}}) \times \frac{(2 - \frac{1}{2^{a_2}})}{2} \\ &= f(\frac{n}{2^{a_2}}) \times \frac{2^{a_2+1} - 1}{2^{a_2+1}}\end{aligned}$$

when $2^{a_2} \parallel n$ due to the Proposition 1.2. In this way, we have

$$\frac{f(\frac{n}{2^{a_2}})}{f(\frac{n}{N})} \leq \frac{2^{a_2+1}}{2^{a_2+1} - 1}.$$

However, we know that $p < 2^{a_2}$ because of $p > 10^8 > 11$ and the Propositions 1.6 and 1.10, where p is the largest prime factor of n . Consequently,

$$\frac{2^{a_2+1}}{2^{a_2+1} - 1} \leq \frac{2 \times p}{2 \times p - 1}$$

since $\frac{x}{x-1}$ decreases when $x \geq 2$ increases. In addition, we know that

$$\frac{2 \times p}{2 \times p - 1} \leq f(p)$$

where we know that $f(p) = \frac{p+1}{p}$ from the Proposition 1.2. Certainly,

$$\begin{aligned}2 \times p^2 &\leq (p+1) \times (2 \times p - 1) \\ &= 2 \times p^2 + 2 \times p - p - 1 \\ &= 2 \times p^2 + p - 1\end{aligned}$$

where this inequality is satisfied for every prime number p . So,

$$\frac{f(\frac{n}{2^{a_2}})}{f(\frac{n}{N})} \leq f(p)$$

where we know that $p \parallel n$ from the Proposition 1.6. Under the assumption of the Riemann Hypothesis, we have that

$$\begin{aligned}e^\gamma &> G(n) \\ &= \frac{f(\frac{n}{p}) \times f(p)}{\log \log n} \\ &\geq \frac{f(\frac{n}{p}) \times f(\frac{n}{2^{a_2}})}{f(\frac{n}{N}) \times \log \log n}\end{aligned}$$

since $f(\dots)$ is multiplicative and as a consequence of the Proposition 1.3. This is equivalent to

$$\frac{f(\frac{n}{p})}{f(\frac{n}{N})} < \frac{e^\gamma}{f(\frac{n}{2^{a_2}})} \times \log \log n.$$

Under the assumption of the Riemann Hypothesis and the Lemma 2.1, we deduce that:

$$\frac{\pi^2}{6.4} \times \prod_{q \leq p} \left(1 + \frac{1}{q}\right) > e^\gamma \times \log \theta(p)$$

which is the same as

$$\frac{\pi^2}{8} \times \prod_{q \leq p} \left(1 + \frac{1}{q}\right) > e^\gamma \times \log((\theta(p))^{0.8}).$$

From the Propositions 1.1 and 1.6, we know that

$$f\left(\frac{n}{2^{a_2}}\right) = \left(\prod_{i=2}^k \frac{q_i}{q_i - 1}\right) \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right)$$

where $q_k = p$ and $q_1 = 2$. We know that

$$\frac{q_i}{q_i - 1} = \frac{q_i + 1}{q_i} \times \frac{q_i^2}{q_i^2 - 1}.$$

Using the previous inequality and the Lemma 2.1, we obtain that

$$\begin{aligned} e^\gamma \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) \times \log((\theta(p))^{0.8}) &< \frac{\pi^2}{8} \times \prod_{q \leq p} \left(1 + \frac{1}{q}\right) \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) \\ &= f\left(\frac{n}{2^{a_2}}\right) \times \frac{3}{2} \times \prod_{q > p} \frac{q^2}{q^2 - 1} \\ &\leq f\left(\frac{n}{2^{a_2}}\right) \times \frac{3}{2} \times e^{\frac{2}{p}} \end{aligned}$$

according to the Proposition 1.9. Taking into account that $p > 10^8 > 3$ and n is superabundant:

$$\frac{\frac{3}{2} \times e^{\frac{2}{p}}}{\log((\theta(p))^{0.8})} > \frac{e^\gamma}{f\left(\frac{n}{2^{a_2}}\right)} \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right).$$

We use the previous inequality to show that

$$\frac{f\left(\frac{n}{p}\right)}{f\left(\frac{n}{N}\right)} \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) < \frac{\frac{3}{2} \times e^{\frac{2}{p}}}{\log((\theta(p))^{0.8})} \times \log \log n.$$

For large enough superabundant number n and $p > 10^8$, then

$$\frac{\frac{3}{2} \times e^{\frac{2}{p}}}{\log((\theta(p))^{0.8})} \times \log \log n \leq \frac{\frac{3}{2} \times e^{\frac{2}{10^8}}}{\log\left(\left(1 - \frac{0.068}{\log 10^8}\right) \times 10^8\right)^{0.8}} \times \log\left(\left(1 + \frac{0.5}{\log 10^8}\right) \times 10^8\right)$$

because of the Propositions 1.4 and 1.6. We obtain that

$$\frac{\frac{3}{2} \times e^{\frac{2}{10^8}}}{\log\left(\left(1 - \frac{0.068}{\log 10^8}\right) \times 10^8\right)^{0.8}} \times \log\left(\left(1 + \frac{0.5}{\log 10^8}\right) \times 10^8\right) < 1.87811.$$

Thus,

$$\frac{f\left(\frac{n}{p}\right)}{f\left(\frac{n}{N}\right)} \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) < 1.87811.$$

For every prime p_j that divides N such that $p_j^{a_j} \parallel N$ and $p_j^{a_j+b_j} \parallel n$ for a_j, b_j two natural numbers, we have that

$$f(p_j^{a_j+b_j}) - f(p_j^{a_j}) \times f(p_j^{b_j}) = -\frac{(p_j^{a_j} - 1) \times (p_j^{b_j} - 1)}{p_j^{a_j+b_j-1} \times (p_j - 1)^2}$$

in the Proposition 1.2. This is equal to

$$\frac{f(p_j^{a_j+b_j})}{f(p_j^{b_j})} = f(p_j^{a_j}) - \frac{(p_j^{a_j} - 1) \times (p_j^{b_j} - 1)}{f(p_j^{b_j}) \times p_j^{a_j+b_j-1} \times (p_j - 1)^2}.$$

Hence,

$$\begin{aligned} \frac{f\left(\frac{n}{p}\right)}{f\left(\frac{n}{N}\right)} \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) &= \prod_j \left(\frac{f(p_j^{a_j+b_j})}{f(p_j^{b_j})}\right) \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) \\ &= \prod_j \left(f(p_j^{a_j}) - \frac{(p_j^{a_j} - 1) \times (p_j^{b_j} - 1)}{f(p_j^{b_j}) \times p_j^{a_j+b_j-1} \times (p_j - 1)^2}\right) \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) \\ &> 1.999 \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) \\ &> 1.999 \times \left(1 - \frac{1}{\log p} \times \left(1 + \frac{1.5}{\log p}\right)\right) \times \frac{1}{\left(1 - \frac{1}{2^{a_2+1}}\right)} \\ &> 1.999 \times \left(1 - \frac{1}{\log p} \times \left(1 + \frac{1.5}{\log p}\right)\right) \\ &> 1.999 \times \left(1 - \frac{1}{\log 10^8} \times \left(1 + \frac{1.5}{\log 10^8}\right)\right) \\ &> 1.88 \\ &> 1.87811 \end{aligned}$$

using the Propositions 1.6 and 1.1 since we know that the expression

$$\frac{(p_j^{a_j} - 1) \times (p_j^{b_j} - 1)}{f(p_j^{b_j}) \times p_j^{a_j+b_j-1} \times (p_j - 1)^2}$$

tends to 0 as b_j tends to infinity for every odd prime p_j where

$$\begin{aligned} \prod_j \left(f(p_j^{a_j}) - \frac{(p_j^{a_j} - 1) \times (p_j^{b_j} - 1)}{f(p_j^{b_j}) \times p_j^{a_j+b_j-1} \times (p_j - 1)^2}\right) &\approx \prod_j (f(p_j^{a_j})) \\ &= f(N) \\ &= 2. \end{aligned}$$

Certainly, the fraction $\frac{f(\frac{n}{p})}{f(\frac{n}{N})}$ gets closer to 2 as long as we take n bigger and bigger. In addition, we note that

$$\begin{aligned} \left(1 - \frac{1}{\log p} \times \left(1 + \frac{1.5}{\log p}\right)\right) &< \prod_{i=1}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) \\ &= \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) \times \left(1 - \frac{1}{2^{a_2+1}}\right) \end{aligned}$$

after taking into account the Proposition 1.6. However,

$$1.87811 < \frac{f(\frac{n}{p})}{f(\frac{n}{N})} \times \prod_{i=2}^k \left(1 - \frac{1}{q_i^{a_i+1}}\right) < 1.87811$$

is a contradiction. By contraposition, the number N does not exist under the assumption of the Riemann Hypothesis. The smallest counterexample N must comply that $N > 10^{1500}$ and therefore, we will always be capable of obtaining a large enough superabundant number n that is multiple of N . Note that, this proof fails for even perfect numbers. \square

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References

- [1] A. Hertlein, Robin's Inequality for New Families of Integers, *Integers* 18, (2018).
- [2] R. Vojak, On numbers satisfying Robin's inequality, properties of the next counterexample and improved specific bounds, arXiv preprint arXiv:2005.09307, (2020).
- [3] G. Robin, Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann, *J. Math. pures appl* 63 (2) (1984) 187–213.
- [4] S. Nazardonyavi, S. Yakubovich, Superabundant numbers, their subsequences and the Riemann hypothesis, arXiv preprint arXiv:1211.2147, (2012).
- [5] P. Solé, M. Planat, Extreme values of the Dedekind ψ function, *Journal of Combinatorics and Number Theory* 3 (1) (2011) 33–38.
- [6] Y. Choie, N. Lichiardopol, P. Moree, P. Solé, On Robin's criterion for the Riemann hypothesis, *Journal de Théorie des Nombres de Bordeaux* 19 (2) (2007) 357–372. doi:doi:10.5802/jtnb.591.
- [7] L. Alaoglu, P. Erdős, On highly composite and similar numbers, *Transactions of the American Mathematical Society* 56 (3) (1944) 448–469. doi:doi:10.2307/1990319.
- [8] G. H. Hardy, E. M. Wright, *An introduction to the theory of numbers*, Oxford University Press, 1979.
- [9] Y. Choie, M. Planat, P. Solé, On Nicolas criterion for the Riemann hypothesis, arXiv preprint arXiv:1012.3613, (2010).
- [10] P. Ochem, M. Rao, Odd perfect numbers are greater than 10^{1500} , *Mathematics of Computation* 81 (279) (2012) 1869–1877. doi:doi:10.1090/S0025-5718-2012-02563-4.
- [11] T. Goto, Y. Ohno, Odd perfect numbers have a prime factor exceeding 10^8 , *Mathematics of Computation* 77 (263) (2008) 1859–1868.
- [12] L. Kozma, Useful Inequalities, http://www.lkozma.net/inequalities_cheat_sheet/ineq.pdf, accessed 4 June 2022 (2022).