On Nicolas Criterion for the Riemann Hypothesis

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September 25, 2023

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Abstract

The Riemann hypothesis is the assertion that all non-trivial zeros have real part $\frac{1}{2}$. It is considered by many to be the most important unsolved problem in pure mathematics. There are several statements equivalent to the famous Riemann hypothesis. In 1983, Nicolas stated that the Riemann hypothesis is true if and only if the inequality $\prod_{q \leq x} \frac{q}{q-1} > e^{\gamma} \cdot \log \theta(x)$ holds for all $x \geq 2$, where $\theta(x)$ is the Chebyshev function, $\gamma \approx 0.57721$ is the Euler-Mascheroni constant and log is the natural logarithm. In this note, using Nicolas criterion, we prove that the Riemann hypothesis is true.

Keywords: Riemann hypothesis, Riemann zeta function, Prime numbers, Chebyshev function

2000 MSC: 11M26, 11A41, 11A25

1. Introduction

The Riemann hypothesis is a conjecture that the Riemann zeta function has its zeros only at the negative even integers and complex numbers with real part $\frac{1}{2}$. It was proposed by Bernhard Riemann (1859). The Riemann hypothesis belongs to the Hilbert's eighth problem on David Hilbert's list of

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twenty-three unsolved problems. This is one of the Clay Mathematics Institute's Millennium Prize Problems. In mathematics, the Chebyshev function $\theta(x)$ is given by

$$\theta(x) = \sum_{q \leq x} \log q$$

with the sum extending over all prime numbers q that are less than or equal to x, where log is the natural logarithm. Leonhard Euler studied the following value of the Riemann zeta function (1734).

Proposition 1.1. *It is known that[1, (1) pp. 1070]:*

$$\zeta(2) = \prod_{k=1}^{\infty} \frac{q_k^2}{q_k^2 - 1} = \frac{\pi^2}{6},$$

where q_k is the kth prime number.

Proposition 1.2. *For $x \geq 3$ we have [2, Lemma 6.4 pp. 370]:*

$$\left(\prod_{q > x} \frac{q^2}{q^2 - 1}\right) \leq \exp\left(\frac{2}{x}\right).$$

We say that $\mathsf{Nicolas}(x)$ holds provided that

$$\prod_{q \leq x} \frac{q}{q-1} > e^{\gamma} \cdot \log \theta(x),$$

where $\gamma \approx 0.57721$ is the Euler-Mascheroni constant. Next, we have the Nicolas Theorem:

Proposition 1.3. $\mathsf{Nicolas}(x)$ *holds for all $x \geq 2$ if and only if the Riemann hypothesis is true [3, Theorem 3 (b) pp. 376].*

In number theory, $\Psi(n) = n \cdot \prod_{q|n} \left(1 + \frac{1}{q}\right)$ is called the Dedekind Ψ function, where $q \mid n$ means the prime q divides n. For $x \geq 2$, a natural number M_x is defined as

$$M_x = \prod_{q \leq x} q.$$

We define $R(n) = \frac{\Psi(n)}{n \cdot \log \log n}$ for $n \geq 3$.

Proposition 1.4. *Unconditionally on Riemann hypothesis, we know that [4, Proposition 3. pp. 3]:*

$$\lim_{x \to \infty} R(M_x) = \frac{e^{\gamma}}{\zeta(2)}.$$

Putting all together yields a proof for the Riemann hypothesis.

2. Central Lemma

The function f was introduced by Nicolas in his seminal paper [3, Theorem 3 pp. 376]:

$$f(x) = e^{\gamma} \cdot \log \theta(x) \cdot \prod_{q \leq x} \left(1 - \frac{1}{q}\right).$$

This is a key Lemma.

Lemma 2.1. *The Riemann hypothesis is true when*

$$\exp\left(\frac{2}{\sqrt{x}}\right) \geq f(x)$$

for large enough x.

Proof. When the Riemann hypothesis is false, then there exists a real number $b < \frac{1}{2}$ for which there are infinitely many natural numbers x such that

$\log f(x) = \Omega_+(x^{-b})$ [3, Theorem 3 (c) pp. 376]. According to the Hardy and Littlewood definition, this would mean that

$$\exists k > 0, \forall y_0 \in \mathbb{N}, \exists y \in \mathbb{N}\ (y > y_0)\colon\ \log f(y) \geq k \cdot y^{-b}.$$

That inequality is equivalent to $\log f(y) \geq \left(k \cdot y^{-b} \cdot \sqrt{y}\right) \cdot \frac{1}{\sqrt{y}}$, but we note that

$$\lim_{y\to\infty} \left(k \cdot y^{-b} \cdot \sqrt{y}\right) = \infty > 2$$

for every possible positive value of k when $b < \frac{1}{2}$. In this way, this implies that

$$\forall y_0 \in \mathbb{N}, \exists y \in \mathbb{N}\ (y > y_0)\colon\ \log f(y) > \frac{2}{\sqrt{y}}.$$

Hence, if the Riemann hypothesis is false, then there are infinitely many natural numbers x such that $\log f(x) > \frac{2}{\sqrt{x}}$. So, if we have

$$\frac{2}{\sqrt{x}} \geq \log f(x)$$

for large enough x, then the Riemann hypothesis cannot be false. By Reductio ad absurdum, the proof is done. $\square$

3. Main Theorem

This is the main theorem.

Theorem 3.1. *The Riemann hypothesis is true.*

Proof. The Riemann hypothesis is true when

$$\exp\left(\frac{2}{\sqrt{x}}\right) \geq f(x)$$

for large enough x by Lemma 2.1. That is equivalent to

$$\exp\left(\frac{2}{\sqrt{x}}\right) \cdot \left(\prod_{q \leq x} \frac{q^2}{q^2-1}\right) \cdot R(M_x) \geq e^{\gamma}.$$

We know that

$$\exp\left(\frac{2}{\sqrt{x}}\right) \cdot \left(\prod_{q \leq x} \frac{q^2}{q^2-1}\right) \gg \zeta(2)$$

where $\gg$ means "much greater than" by Propositions 1.1 and 1.2. Moreover, we know that

$$R(M_x) \sim \frac{e^{\gamma}}{\zeta(2)} \quad when \quad (x \to \infty)$$

by Proposition 1.4. Consequently, the inequality

$$\exp\left(\frac{2}{\sqrt{x}}\right) \cdot \left(\prod_{q \leq x} \frac{q^2}{q^2-1}\right) \cdot R(M_x) \geq e^{\gamma}$$

necessarily holds for large enough x. $\square$

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