# Generalized Craig Interpolants for Stochastic Satisfiability modulo theory problems

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German Research Council, SFB/TR 14

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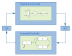
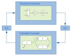
Vienna, Austria, 17.07.2014



# Motivation(1)

**Classical Case:**

**Real-world system**



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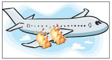

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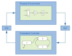

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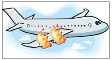

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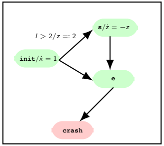


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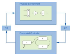

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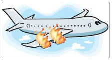

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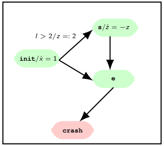


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$\texttt{init}(\vec{s})$
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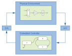
**Mathematical representation**



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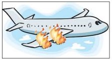

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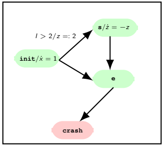


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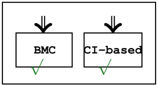


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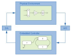

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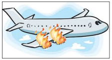
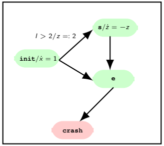
|  | Safety property; no crash | |
|  | Formal Model | |
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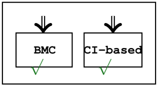
|  | Verification | |

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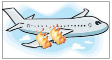
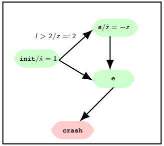
|  | Safety property; no crash | Safety property; Pr(crash) $\leq 10^{-9}$ |
|  | Formal Model | |
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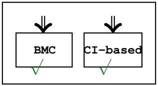
|  | Verification | |

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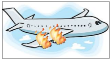

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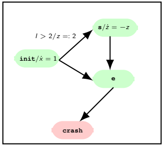


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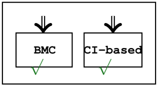


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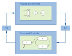

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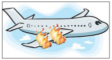
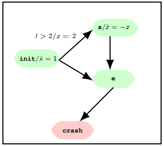
|  | Safety property; no crash | Safety property; Pr(crash) $\leq 10^{-9}$ |
|   | Formal Model | Plus: nondeterministic and probabilistic choices |
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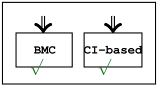
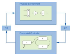
|   | Verification | |



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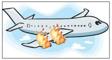
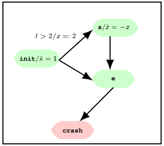
|  | Safety property; no crash | Safety property; Pr(crash) $\leq 10^{-9}$ |
|  | Formal Model | Plus: nondeterministic and probabilistic choices |
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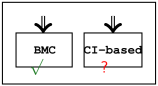
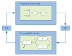
|  | Verification | Verification |







# Motivation(2)

Classical (non-Probabilistic) Case: $TS \models AG\neg p$.


System modelled by a transition system TS.

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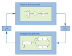
Our Task: verify that the system does not reach unsafe states.



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Explore one step (interpolant $\mathcal{I}_1$) further. $\mathcal{I}_1 \models^{?} AG\neg p$

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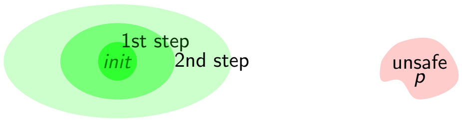


Explore one step ($\mathcal{I}_2$) further. $\mathcal{I}_2 \models^? AG\neg p$.

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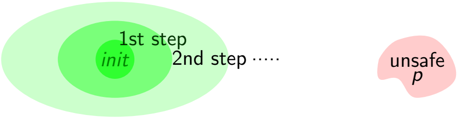


Continue exploring

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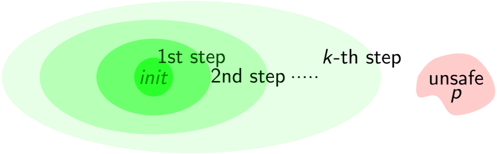


until $\mathcal{I}_k$ stabilizes. $\mathcal{I}_k \models^? AG\neg p$ or $\not\models \mathcal{I}_k \wedge$ *unsafe*.

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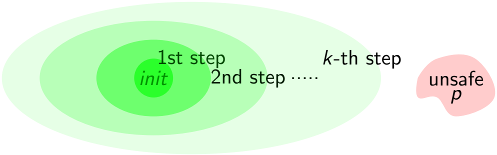


Reachability analysis of *non-probabilistic* finite-state systems based on CI.

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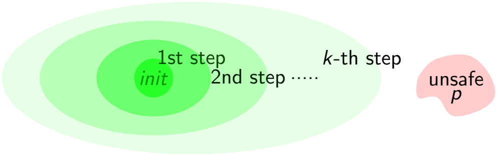


How about to verify that $\mathtt{Pr}(TS \wedge p) \leq \theta$!!

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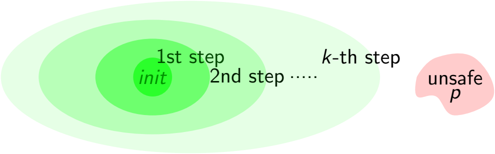
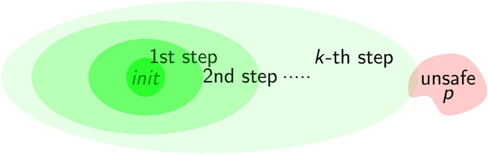



Probabilistic Case: $\mathtt{Pr}(\mathcal{I}_k \wedge p) \leq \theta$



# Outline

- Craig interpolation
- SSMT problems
- Resolution Calculus for SSMT problems
- Generalized Craig interpolation for SSAT and SSMT problems
- Conclusion and future work.

# Craig Interpolation

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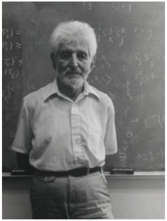


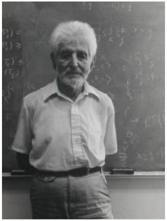
Figure: William Craig, 1957

**Theorem 1 (Craig Interpolation [Cra57])**

*Let $A$ and $B$ be closed FOF. If $A \rightarrow B$ is valid, there exists a formula $\mathcal{I}$ such that:*

- $A \rightarrow \mathcal{I}$
- $\mathcal{I} \rightarrow B$
- $Var(\mathcal{I}) \subseteq Var(A) \cap Var(B)$.

# Craig interpolation



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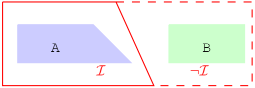


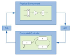

**Example** 1 (Simple Craig interpolant)

*Let* $A = P \wedge Q$, *and* $B = R \rightarrow Q$, *then:*

- $A \rightarrow B$,
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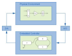

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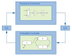

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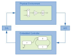

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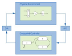

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Craig Interpolations is used as generalizations in:

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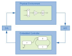
- model checking in particular from BMC, to unbounded model checking [McM03]

$A$ $\neg B$



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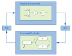

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# Stochastic Satisfiability Modulo Theories

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Stochastic Boolean Satisfiability SSAT [Pap94]

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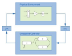
Stochastic Boolean Satisfiability SSAT [Pap94]
= Boolean Satisfiability + Randomized quantifiers



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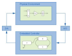

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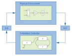
# Stochastic Satisfiability Modulo Theories: Syntax

SSMT formula $\mathcal{Q} : \varphi$

1. prefix $\mathcal{Q}$ of quantified variables
   - $\exists x \in \mathcal{D}_x : \mathcal{D}_x$ is finite. E.g. $\{1, 2, 5, 6\}$
   - $\text{Я} y_{[v_1 \mapsto p_1, \ldots, v_n \mapsto p_n]} : \sum_{i=1}^{n} p_i = 1$. E.g. $\{1 \mapsto 0.5, 2.5 \mapsto 0.21, 7 \mapsto 0.11, 10 \mapsto 0.18\}$
2. SMT formula $\varphi$ (matrix), e.g. $\varphi = (x < 2 \vee \sin(y)) \wedge (a = \texttt{true})\ldots.$



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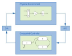

# SSMT: Quantification

$\exists x : \varphi$ I.e., for some value $\varphi$ holds.
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Randomized quantification to describe probabilistic events:

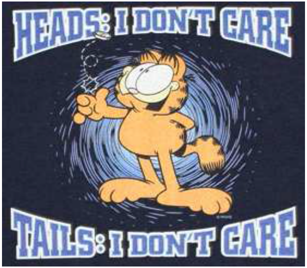


Figure: ᴚ$x_{[head \mapsto 0.5, tail \mapsto 0.5]}$

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Figure: $\text{Я}x_{[2 \mapsto 0.5, 1 \mapsto 0.5, \ldots]} \cdots$

CARL VON OSSIETZKY universität OLDENBURG

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The semantics of an SSMT formula $\Phi$ is given by its maximum probability of satisfaction $Pr(\Phi)$ defined as follows:

$$Pr(\varepsilon : \varphi) = \begin{cases} 0 \text{ } if \text{ } \varphi \text{ } is \text{ } unsatisfiable, \\ 1 \text{ } if \text{ } \varphi \text{ } is \text{ } satisfiable, \end{cases}$$

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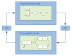
The quantifier-free SMT formula $\Phi$ is called the matrix of $\Phi$



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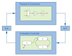

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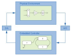

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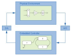


# Example:

$$\Phi = \exists x \in \{2,3,4\}, \text{Я}_{[1 \mapsto 0.2, 2 \mapsto 0.4, 3 \mapsto 0.4]} y \in \{1,2,3\} : (x + y > 3 \vee 2 \cdot y - x > 3) \wedge (x < 4)$$

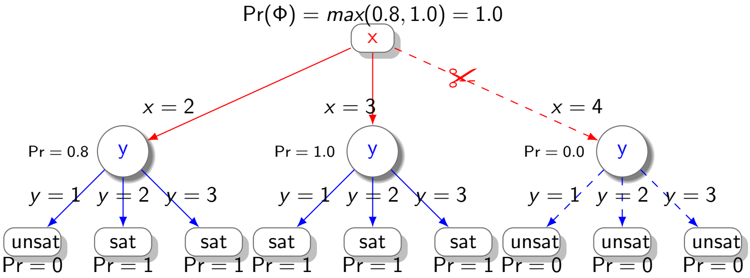


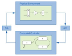

Figure: An example of SSMT formula, the selected part will be traversed and the other part will be pruned from the search space.

# SSMT resolution Calculus

# Resolution Calculus for SAT and SMT

**(Sound and Complete SAT resolution calculus)**

$$\frac{((C_1 \vee x) \wedge (C_2 \vee \neg x))}{(C_1 \vee C_2)} x, \neg x \notin (C_1 \vee C_2) \qquad \text{(SAT-Resolution [Rob65])}$$



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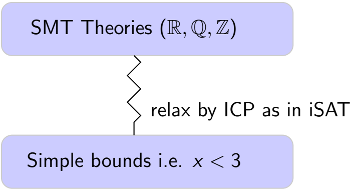

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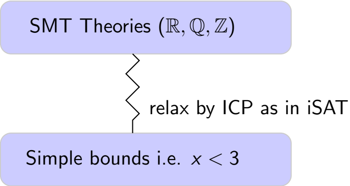


## (Sound and Complete SMT resolution calculus)

$$\frac{(\mathcal{Q} : (C_1 \vee x \sim a) \wedge (C_2 \vee x \sim' b))}{(C_1 \vee C_2)} \mathcal{Q}_x : (x \sim a) \wedge (x \sim' b) \vdash \mathit{false}$$

(SMT-Resolution)

*where* $\sim, \sim' \in \{\leq, <, \geq, >\}$.

# Resolution of SSMT

## (Base case)

$$\frac{(c \in \varphi)}{c^0} \qquad \text{(RR.1)}$$

## Example 1

$$\frac{\exists x \in \{1,5,6\}, \text{Я}_{[4 \mapsto 0.3, 17 \mapsto 0.7]} y : (x \leq 3 \vee y > 10 \vee z > 12) \wedge (x > 5)}{(x \leq 3 \vee y > 10 \vee z > 12)^0, (x > 5)^0} \qquad (1)$$

# Resolution of SSMT

## (Falsification rule)

$$\frac{\begin{pmatrix} c \subseteq \{x \sim a \mid x \in Var(c)\}, \not\models c, \mathcal{Q}(c) = \mathcal{Q}_1 x_1 \ldots \mathcal{Q}_i x_i, \\ \textit{for each } \tau : Var(\varphi) \downarrow_i \rightarrow \mathbb{SB} \textit{ with } \forall x \in Var(\varphi) : \tau(x) \textit{ in } ff_c(x \sim a) : \\ \models \varphi[\tau(x_1)/x_1] \ldots [\tau(x_i)/x_i] \end{pmatrix}}{c^1} \tag{RR.2}$$

## Example 1

$$\frac{\exists x \in \{1, 5, 6\}, \text{Я}_{[4 \mapsto 0.3, 17 \mapsto 0.7]} y : (x \leq 3 \vee y > 10 \vee z > 12) \wedge (x > 5)}{(y \leq 10)^1 \wedge (z \leq 12)^1} \tag{1}$$

# Resolution of SSMT

## (Resolution in case of free variable)

$$\frac{\begin{pmatrix} (x \sim a \vee c_1)^{p_1}, (x \sim' b \vee c_2)^{p_2}, \mathcal{Q}_x \notin \mathcal{Q}, \\ (\exists x : x \sim a \wedge x \sim' b) \vdash False, \not\models (c_1 \vee c_2) \\ p = max(p_1, p_2) \end{pmatrix}}{(c_1 \vee c_2)^p} \quad \text{(RR.3e)}$$

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# Resolution of SSMT

## (Resolution rule between clauses)

$$\frac{\left(\begin{array}{c}(x \sim a \vee c_1)^{p_1}, (x \sim' b \vee c_2)^{p_2}, (\mathcal{Q}_x : x \sim a \wedge x \sim' b \vdash False) \\ \mathcal{Q}_x \in \mathcal{Q}, \not\models (c_1 \vee c_2) \\ p = \begin{cases} max(p_1, p_2) & \text{if } \mathcal{Q} = \exists \\ p_1 \cdot Pr(x \sim' b) + p_2 \cdot Pr(x \sim a) & \text{if } \mathcal{Q} = \reflectbox{R}^{p_x} \end{cases}\end{array}\right)}{(c_1 \vee c_2)^p} \quad \text{(RR.3)}$$

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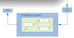

# Generalized Craig Interpolants *for SSMT*

Idea: The same as GCI for SSAT [TF12]; namely:




- we apply the same procedure as in SSAT i.e. adding $\neg S_{A,B}$
- we combine the previous procedure with iSAT reasoning technique i.e. simple bounds.
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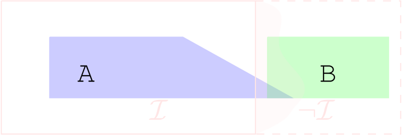


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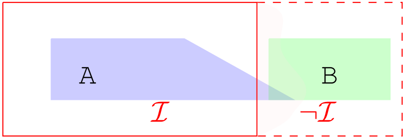


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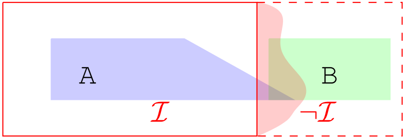


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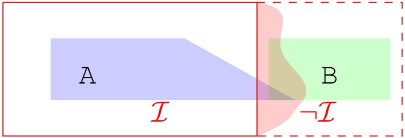


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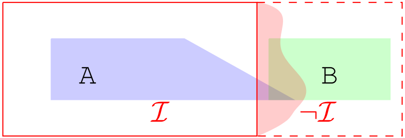


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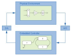

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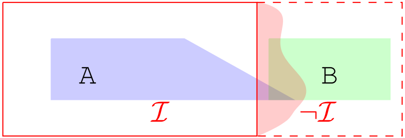


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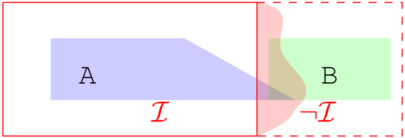


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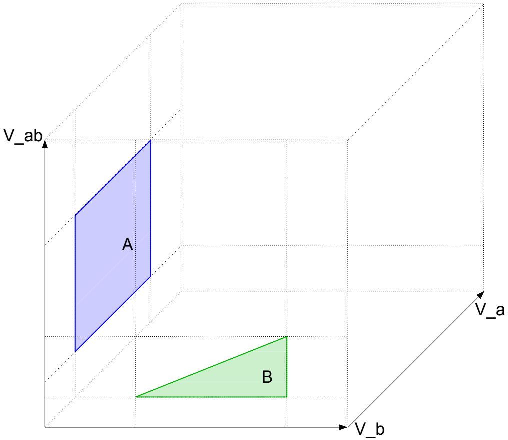


Figure: Generalized Craig interpolation [TF12]

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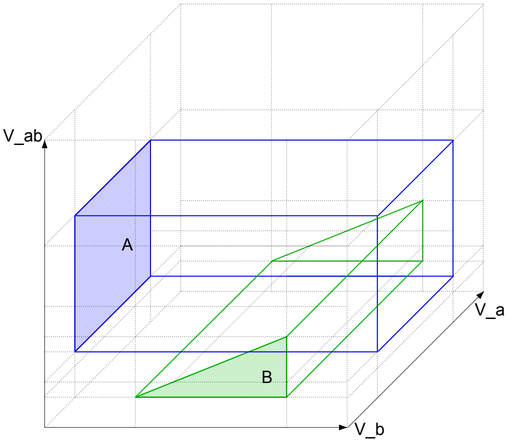


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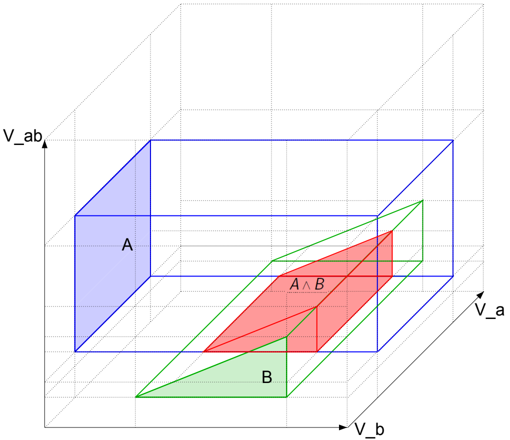


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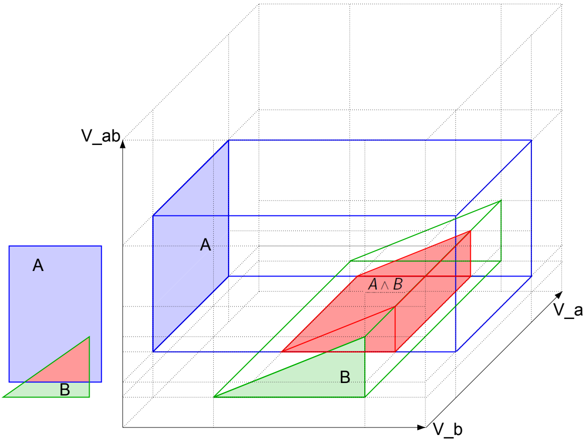


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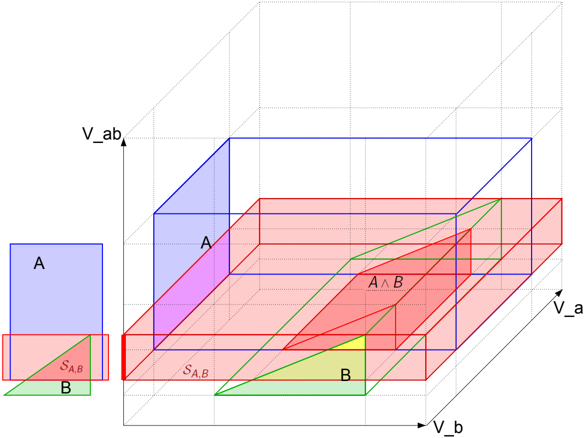


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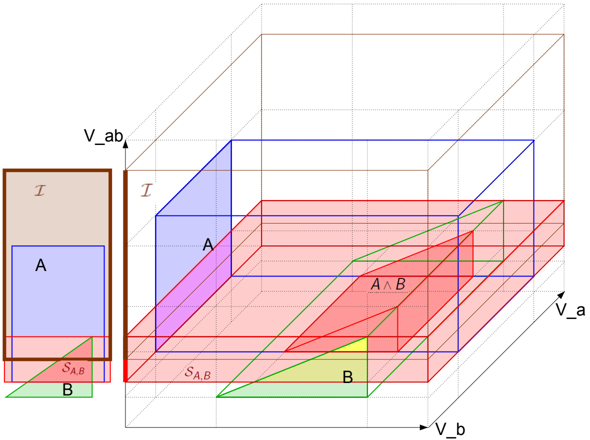



Figure: Generalized Craig interpolation [TF12]

## Definition 1 (Generalized Craig Interpolation–Pudlák extension)

- *Let $A$ and $B$ be some SMT formulae where $V_A := Var(A) \setminus Var(B) = \{a_1, \ldots, a_\alpha\}$, $V_B := Var(B) \setminus Var(A) = \{b_1, \ldots, b_\beta\}$, $V_{A,B} := Var(A) \cap Var(B)$,*
- $A^\exists = \exists a_1, \ldots, a_\alpha : A$, *and*
- $\overline{B}^\forall = \neg \exists b_1, \ldots, b_\beta : B$.

*An SMT formula $\mathcal{I}$ is called a generalized Craig interpolant for $(A, B)$ if and only if the following properties are satisfied:*

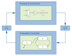
- $Var(\mathcal{I}) \subseteq V_{A,B}$,
- $\models_{\mathcal{L}} (A^\exists \wedge \overline{B}^\forall) \rightarrow \mathcal{I}$,
- $\models_{\mathcal{L}} \mathcal{I} \rightarrow (A^\exists \vee \overline{B}^\forall)$

## Definition cont.

*GCI is computed according to the following rules:*

$$\frac{c \vdash_{R.1} c^p, \quad \mathcal{I} = \begin{cases} False, & c \in A \\ True, & c \in B \end{cases}.}{(c^p, \mathcal{I})} \tag{GR.1}$$

$$\frac{\vdash_{R.2} c^p \quad \mathcal{I} \text{ is any formula over } V_{A,B}}{(c^p, \mathcal{I})} \tag{GR.2}$$



# Definition cont.

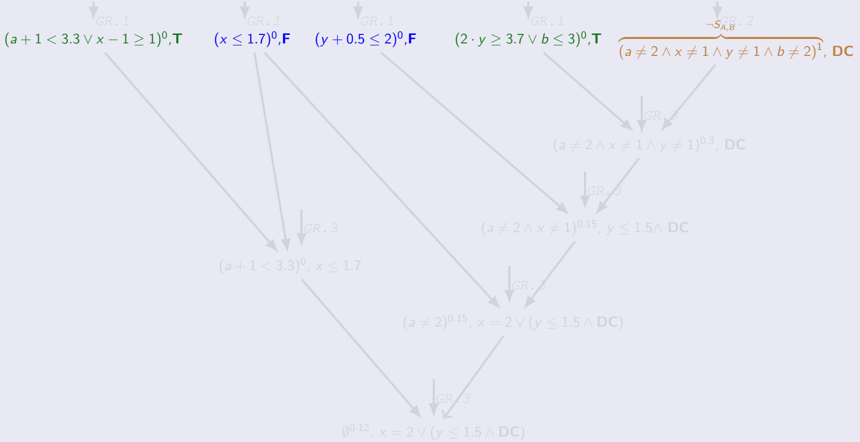
$$
\frac{((x \sim a \vee c_1)^{p_1}, \mathcal{I}_1), ((x \sim b \vee c_2)^{p_2}, \mathcal{I}_2), (x \sim a \wedge x \sim b \vdash false) \quad (x \sim a \vee c_1)^{p_1}, (x \sim b \vee c_2)^{p_2} \vdash_{R.3} (c_1 \vee c_2)^{p}, \quad \mathcal{I} = \begin{cases} \mathcal{I}_1 \vee \mathcal{I}_2 & \text{if } x \in V_A \\ \mathcal{I}_1 \wedge \mathcal{I}_2 & \text{if } x \in V_B \\ (x \sim a \vee \mathcal{I}_1) \wedge (x \sim b \vee \mathcal{I}_2) & \text{if } x \in V_{A,B} \end{cases}}{((c_1 \vee c_2)^{p}, \mathcal{I})} \quad (GR.3)
$$

$$
p = \begin{cases} max(p_1, p_2) & \text{if } \mathcal{Q} = \exists \\ p_1 \cdot Pr(x \sim b) + p_2 \cdot Pr(x \sim a) & \text{if } \mathcal{Q} = \text{Я}^{p_*} \end{cases} \quad (GR.4)
$$

# Example 2

$$\text{Я}.a \in \{2 \mapsto 0.8, 3 \mapsto 0.2\}, \exists.x \in \{1,2\}, \text{Я}.y \in \{1 \mapsto 0.5, 2 \mapsto 0.5\}, \text{Я}.b \in \{2 \mapsto 0.3, 3.2 \mapsto 0.7\}:$$

$$\underbrace{(y + 0.5 \leq 2) \wedge (x \leq 1.7)}_{A} \wedge \underbrace{(a + 1 < 3.3 \vee x - 1 \geq 1) \wedge (2 \cdot y \geq 3.7 \vee b \leq 3)}_{B}$$



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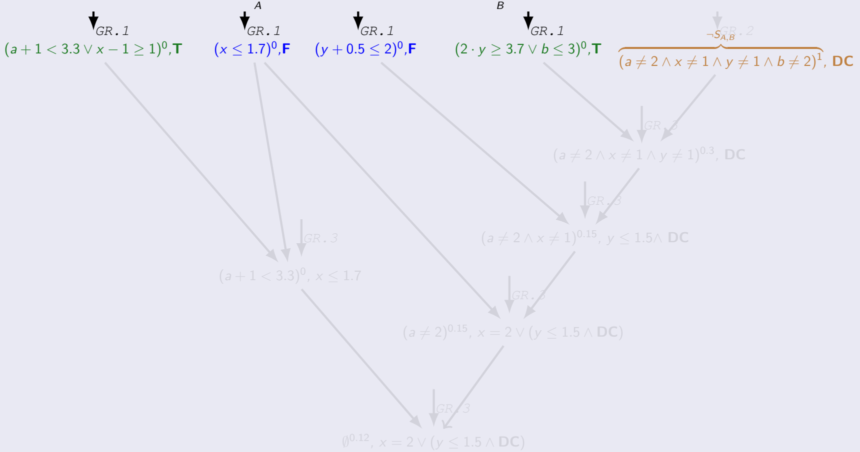

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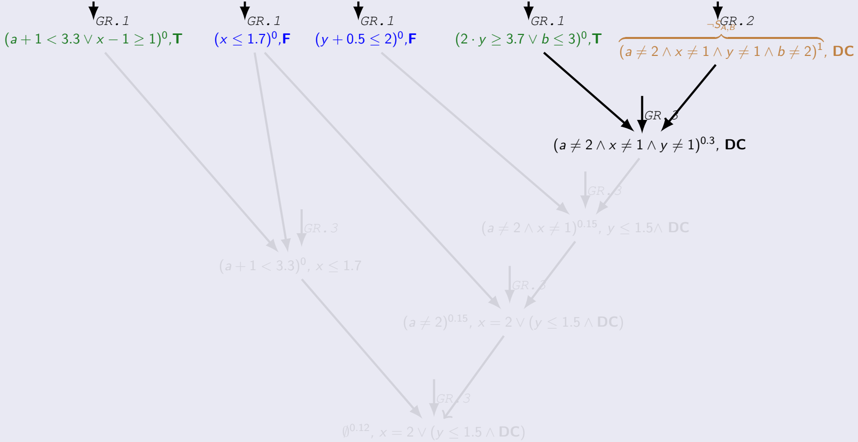

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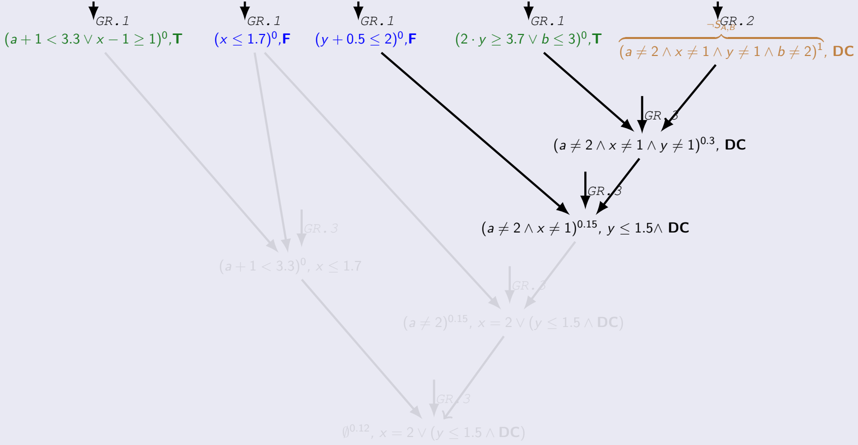

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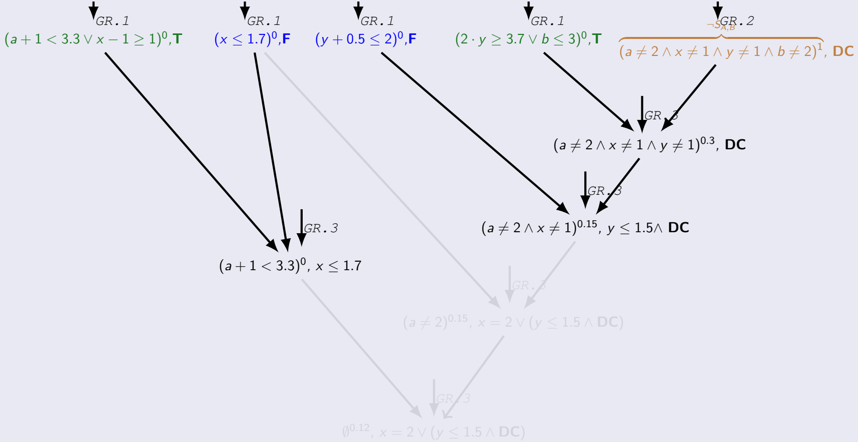

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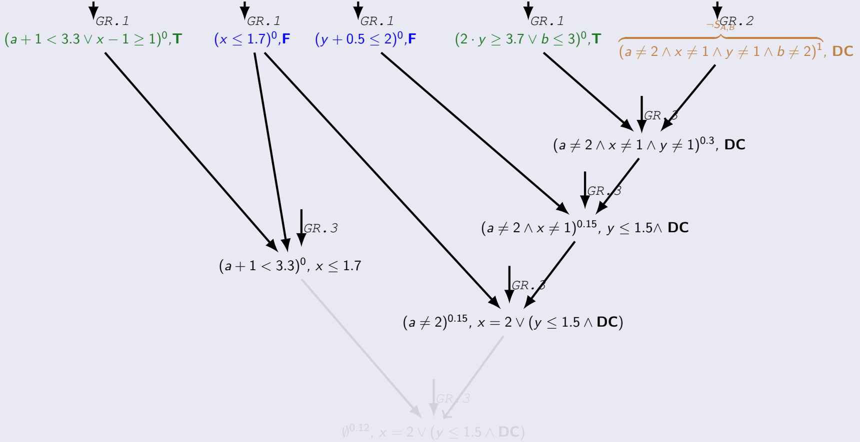

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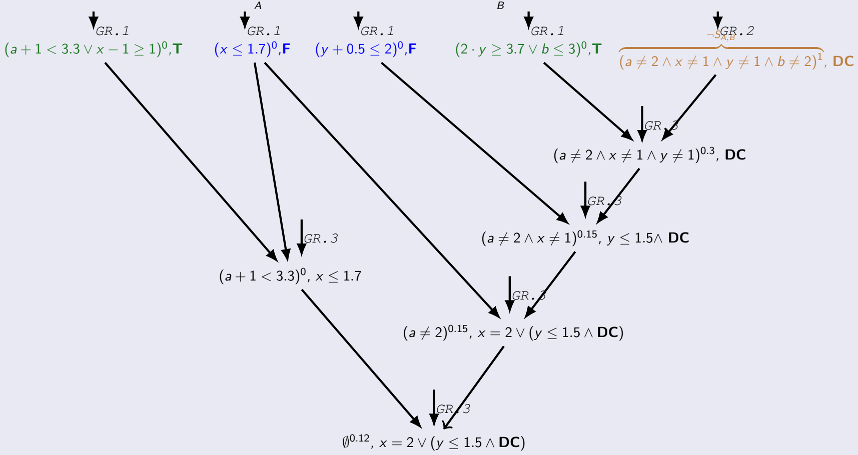

# Conclusion and Future work

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- An approach to compute Craig Interpolant for SSMT problems
- CI is computed regardless of the linearity of a formula.
- All SAT, SSAT, SMT (linear, non-linear, integer and rational) problems are also covered by this approach.
- iSAT interpolants are not simple ones, due to non-linear constraints and ICP ☹.

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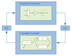

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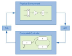

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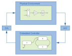

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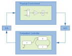
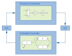


# Future Work

- proper approach to compute $S_{A,B}$ ☹☹.
- slackness of interpolants ☹☺.
- integrate GCI with stochastic CEGAR loop.



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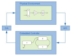

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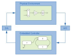

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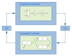

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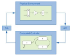

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Thank you for Listening!

Any questions!